Some notes on harmonic and holomorphic functions

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Abstract

These notes are concerned with harmonic and holomorphic functions on Euclidean spaces, where "holomorphic" refers to ordinary complex analysis in dimension 2 and generalizations using quaternions and Clifford algebras in higher dimensions. Among the principal themes are weak solutions, the mean-value property, and subharmonicity.

Contents

1 Euclidean spaces

1.1 Real and complex numbers

Let $\mathbf{R}$ denote the real numbers, equipped with the usual arithmetic operations and ordering. We write $\mathbf{Z}$ and $\mathbf{Z}_+$ for the integers and the positive integers, respectively. The complex numbers are denoted $\mathbf{C}$, and of course a complex number c can be written as $a + i\,b$, where a, b are real numbers and $i^2 = -1$.

Recall that a real number b is said to be an *upper bound* of a subset A of $\mathbf{R}$ if $a \le b$ for all $a \in A$. If A is a set of real numbers and c is a real number, then c is said to be the *least upper bound* or *supremum* of A if c is an upper bound for A and if $c \le b$ for every real number b which is an upper bound for A. It is clear from the definition that the supremum of A is unique if it exists.

Similarly, a real number b' is said to be a *lower bound* for a subset A of $\mathbf{R}$ if $b' \le a$ for every element a of A. If A is a subset of $\mathbf{R}$ and c' is a real number, then c' is said to be the *greatest lower bound* or *infimum* if $b' \le c'$ for every real number b' which is a lower bound for A. Again, it is easy to see from the definition that the infimum is unique if it exists.

The completeness property of the real numbers, with respect to the ordering on the real numbers, states that every nonempty subset of $\mathbf{R}$ which has an upper bound also has a least upper bound. This is equivalent to saying that every nonempty subset of $\mathbf{R}$ which has a lower bound also has a greatest lower bound. The equivalence between the two statements can be shown using multiplication by -1, or by taking the supremum of the set of lower bounds of a subset of $\mathbf{R}$ to get a greatest lower bound and the infimum of the set of upper bounds of a subset of $\mathbf{R}$ to get a least upper bound, as appriopriate.

The supremum and infimum of a subset A of $\mathbf{R}$ are denoted

$$\sup A, \quad \inf A, \tag{1.1}$$

respectively, assuming that they exist. If $f(x)$ is a real-valued function on some nonempty set E, then we may write

$$\sup_{x \in E} f(x), \quad \inf_{x \in E} f(x) \tag{1.2}$$

for the supremum and infimum of $f(x)$ on E, assuming that they exist. More precisely, these are the same as the supremum and infimum of the set

$$\{f(x) : x \in E\} \tag{1.3}$$

of values of f.

If x is a real number, then the *absolute value* of x is denoted $|x|$ and defined to be equal to x when $x \geq 0$ and to $-x$ when $x \leq 0$. One can check that

$$|x + y| \leq |x| + |y| \tag{1.4}$$

for all real numbers x, y, which is the triangle inequality for the absolute value, and that

$$|x\,y| = |x|\,|y|, \tag{1.5}$$

which is to say that the absolute value of a product is equal to the product of the individual absolute values. As a consequence of the triangle inequality, one can check that

$$\Big||x| - |y|\Big| \leq |x - y| \tag{1.6}$$

for all $x, y \in \mathbf{R}$.

The *complex conjugate* of a complex number $c = a + i\,b$, $a, b \in \mathbf{R}$, is denoted $\overline{c}$ and defined by

$$\overline{c} = a - i\,b. \tag{1.7}$$

One can check that

$$\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta} \tag{1.8}$$

and

$$\overline{\alpha\,\beta} = \overline{\alpha}\,\overline{\beta} \tag{1.9}$$

for all $\alpha, \beta \in \mathbf{C}$. If $c \in \mathbf{C}$, $c = a + i\,b$, with $a, b \in \mathbf{R}$, then a and b are called the *real* and *imaginary* parts of c and are denoted $\operatorname{Re} c$, $\operatorname{Im} c$, respectively, and we have that

$$\operatorname{Re} c = \frac{c + \overline{c}}{2}, \quad \operatorname{Im} c = \frac{c - \overline{c}}{2i}. \tag{1.10}$$

If $c = a + i\,b$ is a complex number, $a, b \in \mathbf{R}$, then the *absolute value* or *modulus* is denoted $|c|$ and defined by

$$|c| = \sqrt{a^2 + b^2}. \tag{1.11}$$

Notice that

$$|\operatorname{Re} c|,\ |\operatorname{Im} c| \leq |c| \tag{1.12}$$

and

$$|c|^2 = c\,\overline{c}. \tag{1.13}$$

It follows that the modulus of a product is equal to the product of the moduli, so that

$$|\alpha\,\beta| = |\alpha|\,|\beta| \tag{1.14}$$

for $\alpha, \beta \in \mathbf{C}$.

A basic result states that the triangle inequality holds for complex numbers, i.e.,

$$|\alpha + \beta| \leq |\alpha| + |\beta| \tag{1.15}$$

for all complex numbers α, β. Indeed,

$$\begin{aligned}
|\alpha+\beta|^2 &= |\alpha|^2 + 2\,\mathrm{Re}\,\alpha\,\overline{\beta} + |\beta|^2 \\
&\le |\alpha|^2 + 2|\alpha|\,|\beta| + |\beta|^2 \\
&= (|\alpha| + |\beta|)^2.
\end{aligned} \tag{1.16}$$

As in the case of real numbers, it follows from the triangle inequality that

$$\Big| |\alpha| - |\beta| \Big| \le |\alpha - \beta| \tag{1.17}$$

for $\alpha, \beta \in \mathbf{C}$.

Let n be a positive integer. We write $\mathbf{R}^n$ for the space of n-tuples of real numbers, also known as n-dimensional Euclidean space. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are elements of $\mathbf{R}^n$ and t is a real number, then $x + y$ and $t\,x$ are defined as elements of $\mathbf{R}^n$ using coordinatewise addition and scalar multiplication, so that

$$x + y = (x_1 + y_1, \ldots, x_n + y_n) \tag{1.18}$$

and

$$t\,x = (t\,x_1, \ldots, t\,x_n). \tag{1.19}$$

The *inner product* of $x, y \in \mathbf{R}^n$ is denoted $\langle x, y \rangle$ and defined by

$$\langle x, y \rangle = \sum_{j=1}^n x_j\, y_j. \tag{1.20}$$

Notice that

$$\langle x, y \rangle = \langle y, x \rangle \tag{1.21}$$

for all $x, y \in \mathbf{R}^n$. If $x, x', y \in \mathbf{R}^n$ and $t, t' \in \mathbf{R}$, then

$$\langle t\,x + t'\,x', y \rangle = t\,\langle x, y \rangle + t'\,\langle x', y \rangle, \tag{1.22}$$

and of course there is a similar linearity property in y by symmetry.

The Euclidean norm $|x|$ of $x \in \mathbf{R}^n$ is defined by

$$|x| = \Big(\sum_{j=1}^n x_j^2\Big)^{1/2}. \tag{1.23}$$

This is the same as

$$|x|^2 = \langle x, x \rangle. \tag{1.24}$$

Note that if we identify a complex number c with the element of $\mathbf{R}^2$ given by $(\mathrm{Re}\,c, \mathrm{Im}\,c)$, then the modulus of c as a complex number is the same as the Euclidean norm of the corresponding element of $\mathbf{R}^2$.

An important feature of the inner product on $\mathbf{R}^n$ is the *Cauchy–Schwarz inequality*, which states that

$$|\langle x, y \rangle| \le |x|\,|y| \tag{1.25}$$

for $x, y \in \mathbf{R}^n$. To see this, one can observe that for each real number t,

$$\begin{aligned} 0 &\leq \langle x + t\,y, x + t\,y \rangle \\ &= |x|^2 + 2\,t\,\langle x, y \rangle + t^2\,|y|^2. \end{aligned} \tag{1.26}$$

The Cauchy–Schwarz inequality follows by choosing t appropriately.

For each $x \in \mathbf{R}^n$ and each $t \in \mathbf{R}$ we have that

$$|t\,x| = |t|\,|x|, \tag{1.27}$$

as one can easily check from the definitions. Note that $|t|$ refers to the absolute value of the real number t, while $|t\,x|$, $|x|$ refer to the norms of the vectors $t\,x$, x in $\mathbf{R}^n$. We also again have the triangle inequality

$$|x + y| \leq |x| + |y| \tag{1.28}$$

for $x, y \in \mathbf{R}^n$, since

$$\begin{aligned} |x + y|^2 &= \langle x + y, x + y \rangle \\ &= |x|^2 + 2\,\langle x, y \rangle + |y|^2 \\ &\leq |x|^2 + 2\,|x|\,|y| + |y|^2 \\ &= (|x| + |y|)^2. \end{aligned} \tag{1.29}$$

Two other important features of the Euclidean norm on $\mathbf{R}^n$ are the *polarization identity* and the *parallelogram law*. The polarization identity expresses the inner product in terms of the norm, through the formula

$$\langle x, y \rangle = \frac{1}{2}\Big(|x + y|^2 - |x|^2 - |y|^2\Big) \tag{1.30}$$

for $x, y \in \mathbf{R}^n$. The parallelogram law states that

$$|x + y|^2 + |x - y|^2 = 2\,|x|^2 + 2\,|y|^2, \tag{1.31}$$

which relates the lengths of the sides of a parallelogram to the lengths of the diagonals.

1.2 Linear algebra in $\mathbf{R}^n$

Let n be a positive integer. In this subsection we review matters concerning linear independence, spanning vectors, linear subspaces, and linear mappings in $\mathbf{R}^n$. We begin with a preliminary fact about systems of linear equations.

Suppose that l and m are positive integers, and that for each $j = 1, \ldots, l$ and $k = 1, \ldots, m$ we have a real number $a_{j,k}$. This leads to a system of l homogeneous linear equations in m unknowns, namely the equations

$$\sum_{k=1}^{m} a_{j,k}\, r_k = 0 \tag{1.32}$$

for $j = 1, \ldots, l$. Here $r_1, \ldots, r_m$ are variables which run through the real numbers.

A basic fact is that if $m > l$, then this system of equations has a nontrivial solution. In other words, in this case there are real numbers $r_1, \ldots, r_m$ which satisfy the equations above and which are not all equal to 0. This is not difficult to show, by using the equations to solve for one variable in terms of the other variables and substituting this into the remaining equations to systematically reduce the number of equations.

A collection of vectors $v_1, \ldots, v_m$ in $\mathbf{R}^n$ is said to be *linearly independent* if for any choices of real numbers $r_1, \ldots, r_m$ we have that

$$\sum_{j=1}^{m} r_j \, v_j = 0 \tag{1.33}$$

in $\mathbf{R}^n$ implies that

$$r_1 = \cdots r_m = 0. \tag{1.34}$$

This is equivalent to saying that a vector x in $\mathbf{R}^n$ can be expressed in at most one way as a sum of the form

$$\sum_{j=1}^{m} s_j \, v_j \tag{1.35}$$

for some real numbers $s_1, \ldots, s_m$. From the result mentioned in the previous paragraph it follows that m is necessarily less than or equal to n for this to happen.

Two vectors v, w in $\mathbf{R}^n$ are said to be *orthogonal* if

$$\langle v, w \rangle = 0, \tag{1.36}$$

and in this case we write $v \perp w$. A collection of vectors $v_1, \langle v_m$ is said to be orthogonal if $v_j \perp v_k$ for $1 \le j \, k \le m$, $j \neq k$. If $v_1, \ldots, v_m$ are orthogonal vectors in $\mathbf{R}^n$ such that $|v_j| = 1$ for each j, then $v_1, \ldots, v_m$ is said to be an *orthonormal* collection of vectors in $\mathbf{R}^n$.

A collection of nonzero orthogonal vectors $v_1, \ldots, v_m$ in $\mathbf{R}^n$ is automatically linearly independent. It is enough to check this when $v_1, \ldots, v_m$ are orthonormal, since one can reduce to this case by multiplying the v_j's by nonzero real numbers so that they have norm equal to 1. In this case, if

$$x = \sum_{j=1}^{m} s_j \, v_j \tag{1.37}$$

for some $x \in \mathbf{R}^n$ and real numbers $s_1, \ldots, s_m$, then

$$s_j = \langle x, v_j \rangle \tag{1.38}$$

for each j, so that the coefficients s_j are determined by x.

If $w_1, \ldots, w_l$ is any collection of vectors in $\mathbf{R}^n$, then a *linear combination* of $w_1, \ldots, w_l$ is any vector in $\mathbf{R}^n$ of the form

$$\sum_{k=1}^{l} r_k \, w_k, \tag{1.39}$$

where $r_1, \ldots, r_l$ are real numbers. Thus a finite collection of vectors in $\mathbf{R}^n$ is linearly independent if each element of $\mathbf{R}^n$ which can be written as a linear combination of vectors in the collection can be written as such a linear combination in only one way. A collection of vectors $v_1, \ldots, v_m$ in $\mathbf{R}^m$ is said to be linearly dependent if it is not linearly independent, and this is equivalent to saying that one of the v_j's can be written as a linear combination of the other vectors in the collection.

By a *linear subspace* of $\mathbf{R}^n$ we mean a subset L of $\mathbf{R}^n$ such that $0 \in L$ and $x, y \in L$ implies that $r\,x + s\,y \in L$ for all $r, s \in \mathbf{R}$. If $w_1, \ldots, w_l$ is a collection of vectors in $\mathbf{R}^n$, then the *span* of $w_1, \ldots, w_l$ is denoted

$$\operatorname{span}(w_1, \ldots, w_l) \tag{1.40}$$

and is the subset of $\mathbf{R}^n$ consisting of all linear combinations of $w_1, \ldots, w_l$. It is easy to see that this is always a linear subspace of $\mathbf{R}^n$.

As a basic example, define $e_1, \ldots, e_n \in \mathbf{R}^n$ by saying that e_j has jth coordinate equal to 1 and all other coordinates equal to 0. It is easy to see that $e_1, \ldots, e_n$ is an orthonormal collection of vectors, and is linearly independent in particular. Also, the span of $e_1, \ldots, e_n$ is equal to $\mathbf{R}^n$.

If L is a linear subspace of $\mathbf{R}^n$ and $v_1, \ldots, v_m$ are linearly independent vectors in $\mathbf{R}^n$ whose span is equal to L, then we say that $v_1, \ldots, v_m$ form a *basis* for L. Thus every linear combination of $v_1, \ldots, v_m$ lies in L in this case, and every element of L can be expressed as a linear combination of the v_j's in exactly one way. If the v_j's are also orthogonal, or orthonormal, then we say that they form an *orthogonal basis*, or *orthonormal basis*, of L.

For example, the vectors $e_1, \ldots, e_n$ form an orthonormal basis of $\mathbf{R}^n$. This basis is called the *standard basis* for $\mathbf{R}^n$. Of course this basis has n elements.

In general, if L is a linear subspace of $\mathbf{R}^n$ and $v_1, \ldots, v_m$ is a basis for L, then the *dimension* of L is defined to be m. One can check that the dimension of a subspace is independent of the choice of basis. Indeed, if $w_1, \ldots, w_p$, $v_1, \ldots, v_m$ are vectors in $\mathbf{R}^n$ such that each v_j lies in the span of the w_k's and the v_j's are linearly independent, then $m \le p$.

Let us note that every linear subspace L of $\mathbf{R}^n$ has a basis. In the case where $L = \{0\}$, we interpret this as using the "empty basis", and the dimension is equal to 0. In general one can get a basis by first choosing a nonzero vector in L, and systematically adding vectors to the collection which are not in the span of the vectors already selected, until L is equal to the span of the vectors selected and one gets a basis.

If L is a linear subspace of $\mathbf{R}^n$ and $v_1, \ldots, v_k$ is a collection of linearly independent vectors in L, then one can add vectors to this collection if necessary

to get a basis for L. This follows from the same kind of argument as the one for showing that there is a basis for each linear subspace of $\mathbf{R}^n$. As a consequence, if L_1, L_2 are linear subspaces of $\mathbf{R}^n$ such that $L_1 \subseteq L_2$, then the dimension of L_1 is less than or equal to the dimension of L_2.

Now suppose that L is a linear subspace of $\mathbf{R}^n$ which is equal to the span of a collection of vectors $w_1, \ldots, w_m$. In this case the dimension of L is less than or equal to m. Indeed, either $w_1, \ldots, w_m$ are linearly independent, and hence a basis of L, or one can remove vectors from this collection without changing the span to get a subcollection which is a basis for L.

Suppose that L_1, L_2 are linear subspaces of $\mathbf{R}^n$ such that

$$L_1 \cap L_2 = \{0\}. \tag{1.41}$$

Given a collection of linearly independent vectors in L_1 and another collection of linearly independent vectors in L_2, one can combine them to get a collection of vectors in $\mathbf{R}^n$ which is also linearly independent. Thus the sum of the dimensions of L_1 and L_2 is less than or equal to n.

For any two nonempty subsets A_1, A_2 of $\mathbf{R}^n$, put

$$A_1 + A_2 = \{x \in \mathbf{R}^n : x = a_1 + a_2 \text{ for some } a_1 \in A_1, a_2 \in A_2\}. \tag{1.42}$$

If A_1, A_2 are linear subspaces of $\mathbf{R}^n$, then so is the sum $A_1 + A_2$. If A_1, A_2 are linear subspaces of $\mathbf{R}^n$ such that $A_1 + A_2 = \mathbf{R}^n$, then one can check that the sum of the dimensions of A_1, A_2 is greater than or equal to n.

By a *linear mapping* on $\mathbf{R}^n$ we mean a mapping T from $\mathbf{R}^n$ to itself such that

$$T(x + y) = T(x) + T(y) \tag{1.43}$$

for all $x, y \in \mathbf{R}^n$, and

$$T(r\,x) = r\,T(x) \tag{1.44}$$

for all $r \in \mathbf{R}$ and $x \in \mathbf{R}^n$. In particular,

$$T(0) = 0. \tag{1.45}$$

The identity mapping, which takes x to itself for each $x \in \mathbf{R}^n$, is denoted I.

If T_1, T_2 are linear mappings on $\mathbf{R}^n$, and r_1, r_2 are real numbers, then the linear combination $r_1\,T_1 + r_2\,T_2$ is defined as a linear mapping on $\mathbf{R}^n$, which sends $x \in \mathbf{R}^n$ to $r_1\,T_1(x) + r_2\,T_2(x)$. The composition of T_1, T_2 is denoted $T_1 \circ T_2$ and defined by

$$(T_1 \circ T_2)(x) = T_1(T_2(x)) \tag{1.46}$$

for $x \in \mathbf{R}^n$, and is also a linear mapping on $\mathbf{R}^n$. Of course if T is a linear mapping on $\mathbf{R}^n$, then

$$T \circ I = I \circ T = T. \tag{1.47}$$

Associated to a linear mapping T on $\mathbf{R}^n$ we have a $n \times n$ matrix $(t_{j,k})$ of real numbers, given by

$$t_{j,k} = \langle T(e_k), e_j \rangle, \tag{1.48}$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbf{R}^n$. This is equivalent to saying that

$$T(e_k) = \sum_{j=1}^{n} t_{j,k}\, e_j. \tag{1.49}$$

Conversely, if $(t_{j,k})$ is an $n \times n$ matrix of real numbers, then there is a unique linear mapping T on $\mathbf{R}^n$ associated to this matrix in this manner.

The matrix associated to the identity operator I is the Kronecker delta matrix $(\delta_{j,k})$,

$$\begin{aligned} \delta_{j,k} &= 1 \quad \text{when } j = k \\ &= 0 \quad \text{when } j \neq k. \end{aligned} \tag{1.50}$$

If T_1, T_2 are linear mappings on $\mathbf{R}^n$ and $(t^1_{j,k})$ and $(t^2_{j,k})$ are the $n \times n$ matrices associated to T_1, T_2, respectively, then the matrix associated to $r_1\, T_1 + r_2\, T_2$ is given by $(r_1\, t^1_{j,k} + r_2\, t^2_{j,k})$. The $n \times n$ matrix $(t^3_{j,m})$ associated to the composition $T_3 = T_1 \circ T_2$ is given by

$$t^3_{j,m} = \sum_{l=1^n} t^1_{j,l}\, t^2_{l,m}, \tag{1.51}$$

which is the usual notion of "matrix multiplication".

If L is any linear subspace of $\mathbf{R}^n$, then the *orthogonal complement* of L in $\mathbf{R}^n$ is denoted $L^\perp$ and defined by

$$L^\perp = \{w \in \mathbf{R}^n : w \perp v \text{ for all } v \in L\}. \tag{1.52}$$

It is easy to see that $L^\perp$ is also a linear subspace of $\mathbf{R}^n$. Furthermore,

$$L \cap L^\perp = \{0\}. \tag{1.53}$$

Suppose that L is a linear subspace of $\mathbf{R}^n$ and that $v_1, \ldots, v_m$ is an orthonormal basis for L. Define a linear mapping P_L from $\mathbf{R}^n$ to itself by

$$P_L(x) = \sum_{j=1}^{m} \langle x, v_j \rangle\, v_j. \tag{1.54}$$

This is called the *orthogonal projection of* $\mathbf{R}^n$ *onto* L.

One can check that

$$P_L(x) \in L \quad \text{and} \quad x - P_L(x) \in L^\perp \tag{1.55}$$

for each $x \in \mathbf{R}^n$. Also, if y_1 and y_2 are two vectors in L such that $x - y_1, x - y_2 \in L^\perp$, then $y_1 = y_2$. In other words, $P_L(x)$ is uniquely determined by the properties mentioned above, and in particular $P_L(x)$ does not depend on the choice of the orthonormal basis $v_1, \ldots, v_m$ for L.

If $w_1, \ldots, w_k$ is an orthonormal collection of vectors in $\mathbf{R}^n$ and u is another vector in $\mathbf{R}^n$ which does not lie in $\text{span}(w_1, \ldots, w_k)$, then one can use the

orthogonal projection onto $\mathrm{span}(w_1, \ldots, w_k)$ as just described to define a vector w_{k+1} in $\mathbf{R}^n$ such that

$$w_1, \ldots, w_k, w_{k+1} \tag{1.56}$$

is an orthonormal collection of vectors in $\mathbf{R}^n$ and

$$\mathrm{span}(w_1, \ldots, w_k, u) = \mathrm{span}(w_1, \ldots, w_k, w_{k+1}). \tag{1.57}$$

Using this one can show that every linear subspace of $\mathbf{R}^n$ has an orthonormal basis. As a result, for each linear subspace of $\mathbf{R}^n$ there is an orthogonal projection of $\mathbf{R}^n$ onto that subspace.

If L is a linear subspace of $\mathbf{R}^n$, then

$$L \cap L^\perp = \{0\} \quad \text{and} \quad L + L^\perp = \mathbf{R}^n, \tag{1.58}$$

and in particular the sum of the dimensions of L and $L^\perp$ is equal to n. Let us also observe that

$$(L^\perp)^\perp = L. \tag{1.59}$$

Indeed, $L \subseteq (L^\perp)^\perp$ follows directly from the definition of the orthogonal complement, and conversely if $x \in (L^\perp)^\perp$, then $x - P_L(x)$ lies in both $L^\perp$ and $(L^\perp)^\perp$, and hence is equal to 0, so that $x = P_L(x) \in L$.

By a *bilinear form* we mean a function $B(x, y)$ on $\mathbf{R}^n \times \mathbf{R}^n$ with values in the real numbers which is linear in each of x and y, i.e.,

$$B(r\,x + r'\,x', y) = r\,B(x, y) + r'\,B(x', y) \tag{1.60}$$

for all $r, r' \in \mathbf{R}$ and all $x, x', y \in \mathbf{R}^n$, and

$$B(x, s\,y + s'\,y') = s\,B(x, y) + s'\,B(x, y') \tag{1.61}$$

for all $s, s' \in \mathbf{R}$ and all $x, y, y' \in \mathbf{R}^n$. If T is a linear mapping on $\mathbf{R}^n$, then

$$B(x, y) = \langle T(x), y \rangle \tag{1.62}$$

defines a bilinear form on $\mathbf{R}^n$. Conversely, one can check that every bilinear form on $\mathbf{R}^n$ arises in this manner.

If $B(x, y)$ is a bilinear form on $\mathbf{R}^n$, then we can get a new bilinear form $B^*(x, y)$ simply by interchanging the variables, i.e.,

$$B^*(x, y) = B(y, x). \tag{1.63}$$

If T is a linear transformation on $\mathbf{R}^n$, then there is a unique linear transformation T^* on $\mathbf{R}^n$, called the *adjoint* of T, such that

$$\langle T^*(x), y \rangle = \langle x, T(y) \rangle. \tag{1.64}$$

If $(t_{j,k})$, $(t^*_{j,k})$ are the matrices associated to T, T^*, respectively, then

$$t^*_{j,k} = t_{k,j} \tag{1.65}$$

for $j, k = 1, \ldots, n$.

If r_1, r_2 are real numbers and T_1, T_2 are linear mappings on $\mathbf{R}^n$, then

$$(r_1\, T_2 + r_2\, T_2)^* = r_1\, T_1^* + r_2\, T_2^*. \tag{1.66}$$

Also,

$$(T_1 \circ T_2)^* = T_2^* \circ T_1^*. \tag{1.67}$$

For any linear transformation T on $\mathbf{R}^n$,

$$(T^*)^* = T. \tag{1.68}$$

A bilinear form $B(x, y)$ on $\mathbf{R}^n$ is said to be *symmetric* if $B^* = B$, a linear mapping T on $\mathbf{R}^n$ is said to be *symmetric* if $T^* = T$, and an $n \times n$ matrix $(t_{j,k})$ is said to be *symmetric* if $t_{j,k} = t_{k,j}$. These conditions are related in the obvious way, so that a linear transformation is symmetric if and only if the corresponding bilinear form is symmetric, and this holds if and only if the associated matrix is symmetric. An $n \times n$ matrix $(t_{j,k})$ is said to be *diagonal* if $t_{j,k} = 0$ when $j \neq k$, which clearly implies that the matrix is symmetric.

It is easy to see that the adjoint of the identity mapping is equal to itself. If L is a linear subspace of $\mathbf{R}^n$, then the orthogonal projection P_L of $\mathbf{R}^n$ onto L is symmetric. For if $x, y \in \mathbf{R}^n$, then

$$\langle P_L(x), y\rangle = \langle P_L(x), P_L(y)\rangle = \langle x, P_L(y)\rangle. \tag{1.69}$$

Let T be a linear mapping on $\mathbf{R}^n$. The *kernel* of T is defined to be the set of $x \in \mathbf{R}^n$ such that $T(x) = 0$, and the *image* is defined to be the set of $y \in \mathbf{R}^n$ such that $y = T(x)$ for some $x \in \mathbf{R}^n$. It is easy to see that the kernel and image of T are linear subspaces of $\mathbf{R}^n$.

It is a simple exercise to show that a linear mapping T on $\mathbf{R}^n$ is one-to-one, which means that $T(x') = T(x)$ implies $x' = x$ for $x, x' \in \mathbf{R}^n$, if and only if the kernel of T is trivial, in the sense that it is equal to $\{0\}$. One can always restrict a linear mapping T to the orthogonal complement of its kernel, and on this subspace of $\mathbf{R}^n$ T is one-to-one. Using this one can check that if $v_1, \ldots, v_m$ is a basis for the orthogonal complement of the kernel of T, then $T(v_1), \ldots, T(v_m)$ is a basis for the image of T.

As a result, the dimension of the orthogonal complement of the kernel of T is equal to the dimension of the image of T. Hence the sum of the dimensions of the kernel of T and the image of T is equal to n. This implies in turn that T is one-to-one if and only if T maps $\mathbf{R}^n$ onto $\mathbf{R}^n$, which is to say that the image of T is all of $\mathbf{R}^n$.

Let us note as well that if $y \in \mathbf{R}^n$ lies in the kernel of the adjoint T^* of a linear mapping T on $\mathbf{R}^n$, so that $T^*(y) = 0$, then y lies in the orthogonal complement of the image of T. Conversely, if $y \in \mathbf{R}^n$ is in the orthogonal complement of the image of T, so that

$$\langle y, T(x)\rangle = 0 \tag{1.70}$$

for all $x \in \mathbf{R}^n$, then

$$\langle T^*(y), x \rangle = 0 \tag{1.71}$$

for all $x \in \mathbf{R}^n$, which implies that $T^*(y) = 0$. In short, the kernel of T^* is equal to the orthogonal complement of the image of T, and thus the dimensions of the kernels of T and T^* are equal to each other.

If T is a linear transformation on $\mathbf{R}^n$ and L is a linear subspace of $\mathbf{R}^n$, then L is said to be *invariant* under T if $T(L) \subseteq L$, which is to say that $T(x) \in L$ for all $x \in L$. One can check that if L is invariant under T, then the orthogonal complement $L^\perp$ of L is invariant under the adjoint T^* of T. As a special case, if T is a symmetric linear transformation on $\mathbf{R}^n$ and L is a linear subspace of $\mathbf{R}^n$ which is invariant under L, then $L^\perp$ is also invariant under T.

A linear transformation T on $\mathbf{R}^n$ is said to be *invertible* if there is another linear transformation S on $\mathbf{R}^n$ such that

$$S \circ T = T \circ S = I. \tag{1.72}$$

In this case S is called the *inverse* and is denoted T^{-1}. It is easy to see that the inverse is unique when it exists, and that if T_1, T_2 are invertible linear transformations on $\mathbf{R}^n$, then $T_1 \circ T_2$ is also invertible, with

$$(T_1 \circ T_2)^{-1} = T_2^{-1} \circ T_1^{-1}. \tag{1.73}$$

If T is a linear mapping on $\mathbf{R}^n$ which is invertible simply as a mapping, which is to say that there is a mapping S from $\mathbf{R}^n$ to itself such that $S \circ T = T \circ S = I$, then it is easy to see that S is linear, so that T is invertible as a linear mapping. Of course T is invertible as a mapping on $\mathbf{R}^n$ if it is one-to-one and maps $\mathbf{R}^n$ onto itself. For linear mapping these two conditions are equivalent to each other, as noted before.

A linear transformation T on $\mathbf{R}^n$ is said to be an *orthogonal transformation* if

$$\langle T(x), T(y) \rangle = \langle x, y \rangle \tag{1.74}$$

for all $x, y \in \mathbf{R}^n$. Because of the polarization identity, this is equivalent to saying that

$$|T(x)| = |x| \tag{1.75}$$

for all $x \in \mathbf{R}^n$. One can check that this is also equivalent to saying that T is invertible, and that

$$T^{-1} = T^*. \tag{1.76}$$

If T is a linear mapping on $\mathbf{R}^n$, then the *norm* of T is denoted $\|T\|$ and defined by

$$\|T\| = \sup\{|T(x)| : x \in \mathbf{R}^n, |x| = 1\}. \tag{1.77}$$

It is easy to check that the nonnegative real numbers $|T(x)|$, $x \in \mathbf{R}^n$, $|x| = 1$, are bounded from above, so that the supremum in this definition makes sense. To put it another way, the norm $\|T\|$ of T is the nonnegative real number such that

$$|T(x)| \le \|T\| \, |x| \tag{1.78}$$

for all $x \in \mathbf{R}^n$, and which is as small as possible.

Clearly $\|T\| = 0$ if and only if $T = 0$, and the norm of the identity operator is equal to 1. One can check that

$$\|r\,T\| = |r|\,\|T\| \tag{1.79}$$

for every real number r and every linear operator T on $\mathbf{R}^n$, and that

$$\|T_1 + T_2\| \le \|T_1\| + \|T_2\| \tag{1.80}$$

for all linear mappings T_1, T_2 on $\mathbf{R}^n$. Moreover,

$$\|T_1 \circ T_2\| \le \|T_1\|\,\|T_2\| \tag{1.81}$$

for all linear mappings T_1, T_2 on $\mathbf{R}^n$.

An equivalent definition of the norm of a linear transformation T on $\mathbf{R}^n$ is

$$\|T\| = \sup\{|\langle T(x), y\rangle| : x, y \in \mathbf{R}^n, |x| = |y| = 1\}. \tag{1.82}$$

Indeed, the right side is less than or equal to $\|T\|$ because

$$|\langle T(x), y\rangle| \le \|T\| \tag{1.83}$$

when $x, y \in \mathbf{R}^n$, $|x| = |y| = 1$, by the Cauchy–Schwarz inequality. To get the reverse inequality, one can consider the case where y is a multiple of $T(x)$.

From this equivalent definition of the norm it follows that

$$\|T^*\| = \|T\| \tag{1.84}$$

for any linear mapping on $\mathbf{R}^n$. Another basic fact is the "C^* identity"

$$\|T^* \circ T\| = \|T\|^2. \tag{1.85}$$

This can be checked using the simple formula

$$\langle (T^* \circ T)(x), y\rangle = \langle T(x), T(y)\rangle. \tag{1.86}$$

If L is a linear subspace of $\mathbf{R}^n$ which contains a nonzero element, and P_L is the orthogonal projection of $\mathbf{R}^n$ onto L, then

$$\|P_L\| = 1. \tag{1.87}$$

Indeed, if x, y are vectors in $\mathbf{R}^n$ such that $x \perp y$, then

$$|x + y|^2 = |x|^2 + |y|^2, \tag{1.88}$$

and thus

$$|P_L(u)|^2 \le |P_L(u)|^2 + |u - P_L(u)|^2 = |u|^2 \tag{1.89}$$

for all $u \in \mathbf{R}^n$. This implies that $\|P_L\| \le 1$, and the reverse inequalities hold because $P_L(u) = u$ when $u \in L$.

If T is an orthogonal linear transformation on $\mathbf{R}^n$, then of course

$$\|T\| = 1. \tag{1.90}$$

More precisely, an invertible linear mapping T on $\mathbf{R}^n$ is an orthogonal transformation if and only if

$$\|T\| = \|T^{-1}\| = 1. \tag{1.91}$$

This is easy to check from the definitions.

A symmetric linear transformation T on $\mathbf{R}^n$ is said to be *nonnegative* if

$$\langle T(x), x \rangle \geq 0 \tag{1.92}$$

for all $x \in \mathbf{R}^n$. More generally, if A, B are symmetric linear transformations on $\mathbf{R}^n$, then we say that A is less than or equal to B, written

$$A \leq B, \tag{1.93}$$

if $B - A$ is nonnegative. For any symmetric linear transformation T on $\mathbf{R}^n$, we have that

$$-\|T\|\, I \leq T \leq \|T\|\, I. \tag{1.94}$$

If T is a nonnegative symmetric linear operator on $\mathbf{R}^n$, then

$$|\langle T(x), y \rangle| \leq \langle T(x), x \rangle^{1/2} \, \langle T(y), y \rangle^{1/2}. \tag{1.95}$$

This can be checked in the same manner as for the Cauchy–Schwarz inequality. In particular, $T(x) = 0$ when $\langle T(x), x \rangle = 0$.

If T is a symmetric linear transformation on $\mathbf{R}^n$, then

$$\langle T(x), x \rangle > 0 \tag{1.96}$$

for all $x \in \mathbf{R}^n$ with $x \neq 0$ if and only if T is invertible. Indeed, if this condition holds, then the kernel of T contains only the zero vector, and T is invertible. Conversely, if T is invertible, then $T(x) \neq 0$ for all $x \in \mathbf{R}^n$, and hence the inner product of $T(x)$ and x is nonzero by the result mentioned in the previous paragraph.

1.3 Sequences and series

Fix a positive integer n, and let $\{x_j\}_{j=1}^\infty$ be a sequence of points in $\mathbf{R}^n$. This sequence is said to *converge* to a point $x \in \mathbf{R}^n$ if for every $\epsilon > 0$ there is an integer $N = N(\epsilon)$ so that

$$|x_j - x| < \epsilon \quad \text{for all } j \geq N. \tag{1.97}$$

In this case x is said to be the *limit* of the sequence $\{x_j\}_{j=1}^\infty$, which one can easily show is unique, and one writes

$$\lim_{j \to \infty} x_j = x. \tag{1.98}$$

Sometimes it will be convenient to consider a sequence $\{x_j\}_{j=a}^\infty$ with a different starting point $a \in \mathbf{Z}$, and the same basic notions and results apply just as well. The same definition also applies to sequences of complex numbers, using the standard identification of $\mathbf{C}$ with $\mathbf{R}^2$. In this regard let us note that a sequence $\{x_j\}_j$ in $\mathbf{R}^n$ converges to a point $x \in \mathbf{R}^n$ if and only if for each integer l such that $1 \le l \le n$ the sequence of lth components of the x_j's converges to the lth component of x as a sequence of real numbers, and in particular a sequence $\{z_j\}_j$ of complex numbers converges to a complex number z if and only if the real and imaginary parts of the z_j's converge to the real and imaginary parts of z, respectively.

Suppose that $\{x_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$ are sequences in $\mathbf{R}^n$ which converge to $x, y \in \mathbf{R}^n$, respectively. A standard result which is easy to verify states that $\{x_j + y_j\}_{j=1}^\infty$ converges to $x + y$. In particular this applies to sequences of real and complex numbers.

Now suppose that $\{z_j\}_{j=1}^\infty$, $\{w_j\}_{j=1}^\infty$ are sequences of complex numbers which converge to the complex numbers z, w, respectively. One can then show that the sequence of products $\{z_j \, w_j\}_{j=1}^\infty$ converges to the product $z \, w$ of the limits of the individual sequences. This includes products of sequences of real numbers as a special case, and for a pair of sequences $\{x_j\}_{j=1}^\infty$, $\{y_j\}_{j=1}^\infty$ in $\mathbf{R}^n$ which converge to $x, y \in \mathbf{R}^n$, respectively, one has that the sequence of inner products $\{\langle x_j, y_j \rangle\}_{j=1}^\infty$ converges to the inner product $\langle x, y \rangle$ of the limits.

One way to deal with sequences of products is to consider separately the cases where one of the sequences is constant, and where one of the sequences converges to 0. If one of the sequences tends to 0, then it is sufficient for the other sequence to be bounded in order for the sequence of products to also tend to 0. Let us note as well that if $\{z_j\}_{j=1}^\infty$ is a sequence of nonzero complex numbers which converges to a nonzero complex number z, then the sequence of reciprocals $\{1/z_j\}_{j=1}^\infty$ converges to the reciprocal of the limit, $1/z$.

Sequences of linear transformations on $\mathbf{R}^n$ can be converted into sequences of $n \times n$ matrices, which can then be reinterpreted as sequences in $\mathbf{R}^m$ with $m = n^2$, so that convergence can be defined as before. If $\{S_j\}_{j=1}^\infty$, $\{T_j\}_{j=1}^\infty$ are sequences of linear transformations on $\mathbf{R}^n$ which converge to the linear transformations S, T, respectively, then $\{S_j + T_j\}_{j=1}^\infty$ and $\{S_j \circ T_j\}_{j=1}^\infty$ converge to $S + T$ and $S \circ T$, respectively. If $\{T_j\}_{j=1}^\infty$ is a sequence of invertible linear transformations on $\mathbf{R}^n$ which converges to the invertible linear transformation T, then $\{T_j^{-1}\}_{j=1}^\infty$ converges to T^{-1}, as one can show using "Cramer's rule" for expressing the inverse of a matrix in terms of determinants.

Suppose that $\{x_j\}_{j=1}^\infty$ is a sequence of real numbers. We say that this sequence is *monotone increasing* if $x_{j+1} \ge x_j$ for all j. Similarly, $\{x_j\}_{j=1}^\infty$ is said to be *monotone decreasing* if $x_{j+1} \le x_j$ for all j.

Let $\{x_j\}_{j=1}^\infty$ be a monotone increasing sequence of real numbers which is bounded from above, which is to say that there is a real number b such that $x_j \le b$ for all j. In this case $\{x_j\}_{j=1}^\infty$ converges to a real number x, with

$$x = \sup\{x_j : j \in \mathbf{Z}_+\}. \tag{1.99}$$

In the same way, if $\{y_j\}_{j=1}^\infty$ is a monotone decreasing sequence of real numbers

which is bounded from below, then $\{y_j\}_{j=1}^{\infty}$ converges to $y = \inf_{j \ge 1} y_j$.

A sequence $\{x_j\}_{j=1}^{\infty}$ in $\mathbf{R}^n$ is said to be a *Cauchy sequence* if for every $\epsilon > 0$ there is a positive integer N such that

$$|x_j - x_k| < \epsilon \quad \text{for all } j, k \ge N. \tag{1.100}$$

It is easy to see that a convergent sequence is a Cauchy sequence, and in fact the converse holds too, which is to say that every Cauchy sequence converges. To show this it is enough to consider a Cauchy sequence $\{x_j\}_{j=1}^{\infty}$ of real numbers, and the limit of this sequence is the same as the limit of the monotone increasing sequence $\{x_j'\}_{j=1}^{\infty}$ defined by $x_j' = \inf_{i \ge j} x_i$.

Now let $\sum_{j=0}^{\infty} a_j$ be a series of complex numbers. Associated to this series is the sequence of partial sums

$$\sum_{j=0}^{n} a_j, \tag{1.101}$$

where n runs through the nonnegative integers. If the sequence of partial sums converges to a complex number A, then we say that the series $\sum_{j=0}^{\infty} a_j$ converges to A.

Suppose that $\sum_{j=0}^{\infty} a_j$, $\sum_{j=0}^{\infty} b_j$ are two series of complex numbers which converge to the complex numbers A, B, respectively. If α, β are complex numbers, then the series

$$\sum_{j=0}^{\infty} \alpha\, a_j + \beta\, b_j \tag{1.102}$$

converges to $\alpha\, A + \beta\, B$. This follows from the corresponding statement for sequences, since the partial sums of the new series are equal to the same kind of linear combinations of the partial sums of the original series.

The *Cauchy criterion* for convergence of a series states that a series $\sum_{j=0}^{\infty} a_j$ of complex numbers converges if and only if for every $\epsilon > 0$ there is a positive integer N such that

$$\left| \sum_{j=m}^{n} a_j \right| < \epsilon \tag{1.103}$$

for all m, n such that $n \ge m \ge N$. Indeed, this condition is equivalent to saying that the sequence of partial sums of the series $\sum_{j=0}^{\infty} a_j$ is a Cauchy sequence. A consequence of the Cauchy criterion is the *comparison test* for convergence of a series, which says that if $\sum_{j=0}^{\infty} a_j$ is a series of complex numbers and $\sum_{j=0}^{\infty} b_j$ is a series of nonnegative real numbers which converges such that $|a_j| \le b_j$ for all j, then $\sum_{j=0}^{\infty} a_j$ converges.

Suppose that $\sum_{j=0}^{\infty} b_j$ is a series whose terms are nonnegative real numbers. Then the series converges if and only if the sequence of partial sums is bounded from above. This is because the sequence of partial sums is a monotone increasing sequence of real numbers.

The results mentioned in the preceeding two paragraphs are quite simple, and also quite useful. In short, the question of convergence of a series can often

be reduced to finding an upper bound for some collection of nonnegative real numbers. This is often an easier task.

Let us note a basic necessary condition for a series to converge. Namely, if $\sum_{j=0}^{\infty} a_j$ is a series of complex numbers which converges, then $\{a_j\}_{j=0}^{\infty}$ converges to 0. This is easy to derive from the Cauchy criterion for convergence.

A series $\sum_{j=0}^{\infty} a_n$ *converges absolutely* if $\sum_{j=0}^{\infty} |a_n|$ converges, which implies convergence of the original series in the ordinary sense. Absolute convergence is a more stable kind of convergence. For example, if a series converges absolutely, then one can rearrange the terms and still get a series which converges, and with the same sum.

A famous test for convergence of series is called the *Cauchy Condensation Test*, and it says the following. If $\sum_{j=1}^{\infty} b_j$ is a series of nonnegative real numbers such that the sequence of b_j's is monotone decreasing, then $\sum_{j=1}^{\infty} b_j$ converges if and only if the "condensed" series $\sum_{k=0}^{\infty} 2^k b_{2^k}$ converges. To prove this, one shows that the partial sums of each can be bounded in terms of the other.

Before describing some examples, let us review some auxiliary facts. First, the *binomial theorem* states that for each positive integer n,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j \, y^{n-j}, \tag{1.104}$$

where

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}, \tag{1.105}$$

$m! = 1 \cdot 2 \cdots m$, $0! = 1$. This can be proved by induction, with the $n = 1$ case being trivial, and let us note also that the binomial coefficients $\binom{n}{j}$ are always integers.

Now let a be a positive real number, and observe that

$$(1+a)^n \geq 1 + n\,a \tag{1.106}$$

for all positive integers n, by the binomial theorem. Thus

$$\lim_{n \to \infty} (1+a)^{-n} = 0, \tag{1.107}$$

and hence

$$\lim_{n \to \infty} z^n = 0 \tag{1.108}$$

when z is a complex number with $|z| < 1$. More generally, if z is a complex number with $|z| < 1$ and k is a positive integer, then

$$\lim_{n \to \infty} n^k \, z^n = 0. \tag{1.109}$$

If z is a complex number with $|z| < 1$, then the *geometric series* $\sum_{n=0}^{\infty} z^n$ converges, and

$$\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}. \tag{1.110}$$

Indeed,

$$(1 - z) \sum_{j=0}^{n} = 1 - z^{n+1} \tag{1.111}$$

for each nonnegative integer n, so that

$$\lim_{n \to \infty} \sum_{j=0}^{n} z^j = \lim_{n \to \infty} \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z}. \tag{1.112}$$

Of course the series does not converge when $|z| \ge 1$, since $|z|^n \ge 1$ then for all n, and the terms of the series do not tend to 0.

Now suppose that p is a positive real number, and consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}. \tag{1.113}$$

One can show that this series converges if and only if $p > 1$, using the Cauchy Condensation Test. More precisely, this reduces the question to one for geometric series.

However, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \tag{1.114}$$

converges for all positive real numbers. This can be viewed as a special case of Leibniz' *alternating series test*, which states that a series of the form

$$\sum_{n=1}^{\infty} (-1)^n \, b_n \tag{1.115}$$

converges if $\{b_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of real numbers which converges to 0. One can verify this test using the Cauchy criterion, by estimating the absolute value of sums of the form $\sum_{n=p}^{q} (-1)^n \, b_n$ in terms of the absolute value of the first term b_p.

More generally, a series of the form

$$\sum_{n=1}^{\infty} a_n \, b_n \tag{1.116}$$

converges if the sequence of partial sums $A_p = \sum_{n=1}^{p} a_n$ of the a_n's is bounded and if $\{b_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of real numbers which converges to 0. Indeed, if we set $A_0 = 0$ for convenience, then for each positive integer p,

$$\sum_{n=1}^{p} a_n \, b_n \quad = \quad \sum_{n=1}^{p} (A_n - A_{n-1}) \, b_n \tag{1.117}$$

$$
\begin{aligned}
&= \sum_{n=1}^{p} A_n\, b_n - \sum_{n=1}^{p} A_{n-1}\, b_n \\
&= \sum_{n=1}^{p} A_n\, b_n - \sum_{n=0}^{p-1} A_n\, b_{n+1} \\
&= \sum_{n=1}^{p-1} A_n\, (b_n - b_{n+1}) + A_p\, b_p.
\end{aligned}
$$

One can check that $\sum_{n=1}^{\infty} A_n\, (b_n - b_{n+1})$ converges absolutely, since

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) \tag{1.118}$$

is a convergent "telescoping series" with nonnegative terms, and hence that $\sum_{n=1}^{\infty} a_n\, b_n$ converges.

For example, suppose that z is a complex number such that $|z| = 1$ and $z \neq 1$, and consider $a_n = z^n$. In this case we have that

$$\sum_{n=1}^{p} z^n = \frac{z - z^p}{1 - z}, \tag{1.119}$$

which is a bounded sequence of complex numbers as p runs through the positive integers. Thus we obtain that

$$\sum_{n=1}^{\infty} z^n\, b_n \tag{1.120}$$

converges when z is a complex number such that $|z| = 1$ and $z \neq 1$ and $\{b_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence of real numbers which converges to 0.

1.4 Quarternions and Clifford algebras

The quarternions are denoted $\mathbf{H}$, and a quarternion can be written as $a + b\, i + c\, j + d\, k$, where a, b, c, d are real numbers, and i, j, k satisfy the conditions

$$i^2 = j^2 = k^2 = -1 \tag{1.121}$$

and

$$k = i\, j = -j\, i. \tag{1.122}$$

To be more precise, the quarternions are an algebra over the real numbers, with coordinatewise addition, the real number 1 being the multiplicative identity element for the whole algebra, and real numbers commuting multiplicatively with other quarternions even if multiplication in $\mathbf{H}$ is not commutative in general. Multiplication in $\mathbf{H}$ does satisfy the usual associative and distributive rules, and

$$i\, k = -k\, i = -k, \quad j\, k = k\, j = -i \tag{1.123}$$

in particular.

Suppose that $x = a + b\,i + c\,j + d\,k$ is a quarternion. The *conjugate* of x is denoted x^* and defined by

$$x^* = a - b\,i - c\,k - d\,k. \tag{1.124}$$

If x, y are quarternions, then $(x^*)^* = x$,

$$(x + y)^* = x^* + y^*, \tag{1.125}$$

and

$$(x\,y)^* = y^*\,x^*. \tag{1.126}$$

The *modulus* of a quarternion $x = a+b\,i+c\,j+d\,k$ is denoted $|x|$ and defined by

$$|x| = (a^2 + b^2 + c^2 + d^2)^{1/2}. \tag{1.127}$$

One can check that

$$|x|^2 = x\,x^* = x^*\,x, \tag{1.128}$$

and in particular it follows that a nonzero quarternion x is invertible, with $x^{-1} = |x|^{-2}\,x^*$. Moreover,

$$|x\,y| = |x|\,|y| \tag{1.129}$$

for $x, y \in \mathbf{H}$.

If $x = a + b\,i + c\,j + d\,k$ is a quarternion, with $a, b, c, d \in \mathbf{R}$, as usual, then a is called the *real part* of x. Of course the real part of x is equal to the real part of x^*. One can check that if x, y are quarternions, then the real part of $x\,y$ is equal to the real part of $y\,x$.

We can identify a quarternion $x = a + b\,i + c\,j + d\,k$, $a, b, c, d \in \mathbf{R}$, with the element (a, b, c, d) of $\mathbf{R}^4$. This identification respects the operations of addition and "scalar" multiplication by real numbers on $\mathbf{H}$ and $\mathbf{R}^4$, and the modulus of a quarternion corresponds exactly to the standard Euclidean norm on $\mathbf{R}^4$. If x, y are quarternions, then the real part of $x\,y^*$ is the same as the standard inner product of the cooresponding elements of $\mathbf{R}^4$.

Suppose that α, β are quarternions such that $|\alpha| = |\beta| = 1$, and consider the mapping

$$x \mapsto \alpha\,x\,\beta \tag{1.130}$$

from $\mathbf{H}$ onto itself. If we identify $\mathbf{H}$ with $\mathbf{R}^4$ as in the previous paragraph, then this cooresponds to an orthogonal linear transformation on $\mathbf{R}^4$. This is analogous to the fact that if θ is a complex number such that $|\theta| = 1$, then the mapping

$$z \mapsto \theta\,z \tag{1.131}$$

from $\mathbf{C}$ onto itself corresponds to an orthogonal linear transformation on $\mathbf{R}^2$ using the standard identification of $\mathbf{C}$ with $\mathbf{R}^2$.

A quarternion w is said to be *imaginary* if its real part is equal to 0, which is equivalent to

$$w^* = -w, \tag{1.132}$$

and to

$$w^2 = -|w|^2. \tag{1.133}$$

If α is a quarternion and w is an imaginary quarternion, then $\alpha\, w\, \alpha^*$ is also an imaginary quarternion. If α is a quarternion with $|\alpha| = 1$, then the mapping

$$w \mapsto \alpha\, w\, \alpha^* = \alpha\, w\, \alpha^{-1} \tag{1.134}$$

on imaginary quarternions corresponds, under the natural identification of imaginary quarternions with elements of $\mathbf{R}^3$, to an orthogonal linear transformation on $\mathbf{R}^3$.

Now let n be a positive integer, and let $\mathcal{C}(n)$ denote the *Clifford algebra* with n generators $e_1, \ldots, e_n$. To be more precise, $\mathcal{C}(n)$ is an algebra over the real numbers which contains a copy of the real numbers such that the real number 1 the multiplicative identity element for the whole Clifford algebra and of course real numbers commute multiplicatively with all other elements of the Clifford algebra. The generators $e_1, \ldots, e_n$ satisfy the relations

$$e_j^2 = -1 \tag{1.135}$$

for $j = 1, \ldots, n$ and

$$e_j\, e_k = -e_k\, e_j \tag{1.136}$$

when $j \neq k$.

As an algebra over the real numbers, $\mathcal{C}(n)$ is a vector space over the real numbers. As such it has finite dimension. The basic reason for this is that any product of e_j's can be reduced to one in which each e_l appears at most once.

In fact, the dimension of $\mathcal{C}(n)$ as a vector space over the real numbers is equal to 2^n. For let I be a subset of the set $\{1, \ldots, n\}$, where I consists of

$$i_1 < i_2 < \cdots < i_l \tag{1.137}$$

for some l, $0 \le l \le n$, and let e_I denote the product

$$e_{i_1}\, e_{i_2} \cdots e_{i_l}, \tag{1.138}$$

where this is interpreted as being equal to 1 when $I = \emptyset$. The family of e_I's, where I runs through all subsets of $\{1, \ldots, n\}$, forms a basis for $\mathcal{C}(n)$, which is to say that every element of $\mathcal{C}(n)$ can be expressed in a unique manner as a linear combination of the e_I's.

One can define $\mathcal{C}(n)$ when $n = 0$ to be just the real numbers themselves. When $n = 1$, the Clifford algebra $\mathcal{C}(n)$ is equivalent to the complex numbers. When $n = 2$, the Clifford algebra is equivalent to the quarternions.

Unlike the real numbers, complex numbers, or quarternions, it is not the case in general that every nonzero element of $\mathcal{C}(n)$ has a multiplicative inverse. However, suppose that $x \in \mathcal{C}(n)$ is of the form

$$x = x_0 + \sum_{j=1}^{n} x_j\, e_j, \tag{1.139}$$

where $x_0, x_1, \ldots, x_n \in \mathbf{R}$. If $x \neq 0$, then x is invertible in $\mathcal{C}(n)$, with

$$x^{-1} = \frac{x_0 - \sum_{j=1}^n x_j\, e_j}{x_0^2 + \sum_{j=1}^n x_j^2}. \tag{1.140}$$

2 Harmonic and holomorphic functions, 1

If $x \in \mathbf{R}^n$ and r is a positive real number, then we define $B(x,r)$, the open ball with center x and radius r in $\mathbf{R}^n$ by

$$B(x,r) = \{y \in \mathbf{R}^n : |y - x| < r\}. \tag{2.1}$$

The closed ball with center x and radius r is denoted $\overline{B}(x,r)$ and defined by

$$\overline{B}(x,r) = \{y \in \mathbf{R}^n : |y - x| \le r\}. \tag{2.2}$$

A subset E of $\mathbf{R}^n$ is said to be *bounded* if it is contained in a ball.

A subset U of $\mathbf{R}^n$ is said to be *open* if for every $x \in U$ there is a positive real number r such that

$$B(x,r) \subseteq U. \tag{2.3}$$

The empty set and $\mathbf{R}^n$ itself are automatically open subsets of $\mathbf{R}^n$, and one can check that open balls in $\mathbf{R}^n$ are open subsets, using the triangle inequality for the Euclidean distance $|v - w|$, $v, w \in \mathbf{R}^n$. The *interior* of a subset A of $\mathbf{R}^n$ is denoted A° and is defined to be the set of $x \in A$ for which there is a positive real number r such that $B(x,r) \subseteq A$, and one can check that the interior of A is always an open subset of $\mathbf{R}^n$.

If E is a subset of $\mathbf{R}^n$ and p is a point in $\mathbf{R}^n$, then p is said to be a *limit point* of E if for each positive real number $r > 0$ there is an $x \in E$ such that $x \neq p$ and

$$|x - p| < r. \tag{2.4}$$

A subset E of $\mathbf{R}^n$ is said to be *closed* if every point in $\mathbf{R}^n$ which is a limit point of E is also an element of E. This is equivalent to saying that every sequence $\{x_j\}_{j=1}^\infty$ of points in E which converges to some point $x \in \mathbf{R}^n$ has its limit x in E.

The empty set and $\mathbf{R}^n$ itself are automatically closed subsets of $\mathbf{R}^n$, and one can check that a closed ball in $\mathbf{R}^n$ is a closed subset of $\mathbf{R}^n$. If E is a subset of $\mathbf{R}^n$, then the *closure* of E is denoted $\overline{E}$ and is defined to be the union of E and the set of limit points of E, and it is not difficult to show that the closure of a subset of $\mathbf{R}^n$ is always a closed subset of $\mathbf{R}^n$. If A is a subset of $\mathbf{R}^n$, then A is open if and only if the complement $\mathbf{R}^n \backslash A$ of A in $\mathbf{R}^n$, consisting of the points in $\mathbf{R}^n$ not in A, is a closed subset of $\mathbf{R}^n$.

2.1 Some differential operators

Let U be a nonempty open subset of $\mathbf{R}^n$, and let $f(x)$ be a twice continuously-differentiable real or complex-valued function on U. Put

$$\Delta f = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} f. \tag{2.5}$$

This is called the *Laplacian* of f.

Now suppose that $n = 2$, and identify $\mathbf{R}^2$ with $\mathbf{C}$. Writing $z = x + i\,y$, $x, y \in \mathbf{R}$, for an element of $\mathbf{C}$, if $f(z)$ is a continuously-differentiable complex-valued function on a nonempty open subset U of $\mathbf{C}$, then we put

$$\frac{\partial}{\partial z} f = \frac{1}{2}\left(\frac{\partial}{\partial x} f - i \frac{\partial}{\partial y} f\right) \tag{2.6}$$

and

$$\frac{\partial}{\partial \overline{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} f + i \frac{\partial}{\partial y} f\right). \tag{2.7}$$

If $f(z)$ is twice continuously-differentiable on U, then we have that

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} f = \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} f = \frac{1}{4}\Delta f. \tag{2.8}$$

For general n again, assume that $f(x)$ is a continuously-differentiable function on a nonempty open subset U of $\mathbf{R}^n$ with values in the Clifford algebra $\mathcal{C}(n)$. Define the corresponding *left and right Dirac operators* acting on f by

$$\mathcal{D}_L f = \sum_{j=1}^{n} e_j \, \frac{\partial}{\partial x_j} \, f \tag{2.9}$$

and

$$\mathcal{D}_R f = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \, f \, e_j, \tag{2.10}$$

where $e_1, \ldots, e_n$ are the usual multiplicative basis for $\mathcal{C}(n)$, i.e., we either multiply the first derivatives of f by the e_j's on the left or on the right, respectively. If f is twice continuously-differentiable, then we have that

$$\mathcal{D}_L^2 f = \mathcal{D}_R^2 f = -\Delta f, \tag{2.11}$$

where the Laplacian of f, Δf, is defined for a Clifford-valued function in the same way as before, which amounts to applying the Laplacian for real-valued functions to the components of f.

Let us note that there are some natural variants of this, where for instance one considers functions on $\mathbf{R}^{n+1}$ with values in $\mathcal{C}(n)$, and one uses 1 as well as $e_1, \ldots, e_n$ for defining the first-order differential operators. This type of set-up

is more directly analogous to the one for complex-valued functions on $\mathbf{C}$. For simplicity we shall use the version described in the previous paragraph.

If p is a point in $\mathbf{R}^n$, r is a positive real number, and $f(x)$ is a continuous function on the closed ball $\overline{B}(p,r)$ in $\mathbf{R}^n$, then we write $\mathrm{average}_{\overline{B}(p,r)} f$ for the average of f over $\overline{B}(p,r)$, which is the integral of f over $\overline{B}(p,r)$ divided by the volume of $\overline{B}(p,r)$. For $j,k = 1,\ldots,n$, we have that

$$\mathrm{average}_{\overline{B}(p,r)}(x_j - p_j)^2 = \mathrm{average}_{\overline{B}(p,r)}(x_k - p_k)^2, \tag{2.12}$$

and thus

$$\mathrm{average}_{\overline{B}(p,r)}(x_j - p_j)^2 = \frac{1}{n}\mathrm{average}_{\overline{B}(p,r)}|x-p|^2. \tag{2.13}$$

Also,

$$\mathrm{average}_{\overline{B}(p,r)}|x-p|^2 = \frac{n}{n+2}\, r^2, \tag{2.14}$$

as one can see by reducing to the case where $p = 0$ and using polar coordinates.

Let U be a nonempty open subset of $\mathbf{R}^n$. Suppose that $f(x)$ is a twice continuously-differentiable function on U, and that p is an element of U. One can express the Laplacian of f at p as

$$\Delta f(p) = \lim_{r\to 0} \frac{n+2}{r^2}\Big(\mathrm{average}_{\overline{B}(p,r)} f - f(p)\Big). \tag{2.15}$$

This follows from the Taylor approximation of f at p of degree 2.

A twice continuously-differentiable function f on a nonempty open subset U of $\mathbf{R}^n$ is said to be *harmonic* if

$$\Delta f(p) = 0 \tag{2.16}$$

for all $p \in U$. This is equivalent to saying that

$$\lim_{r\to 0} \frac{1}{r^2}\Big(\mathrm{average}_{\overline{B}(p,r)} f - f(p)\Big) = 0 \tag{2.17}$$

for all $p \in U$. A stronger "mean value property" of harmonic functions will be described in the next subsection.

A continuously-differentiable complex-valued function $f(z)$ on an open subset $U \neq \emptyset$ of $\mathbf{C}$ is said to be *holomorphic* if

$$\frac{\partial}{\partial \overline{z}} f = 0 \tag{2.18}$$

at every point in U. The derivative $f'(z)$ of a holomorphic function $f(z)$ on U is defined by

$$f'(z) = \frac{\partial}{\partial z} f(z). \tag{2.19}$$

If we identify $\mathbf{C}$ with $\mathbf{R}^2$ in the usual way, then the differential of a continuously-differentiable complex-valued function $f(z)$ on U at a point p in U is a linear

mapping from $\mathbf{R}^2$ to itself, and in fact $(\partial/\partial\overline{z})f$ vanishes at p if and only if the differential of f at p is given by multiplication by a complex number on $\mathbf{C}$, and that complex number is equal to $(\partial/\partial z)f$ at p.

A continuously-differentiable $\mathcal{C}(n)$-valued function $f(x)$ on a nonempty open subset U of $\mathbf{R}^n$ is said to be *left* or *right Clifford holomorphic*, respectively, if

$$\mathcal{D}_L f(x) = 0 \tag{2.20}$$

or

$$\mathcal{D}_R f(x) = 0 \tag{2.21}$$

for all $x \in U$, respectively. These two conditions are not equivalent in general, because of noncommutativity, although there are functions which are both left and right Clifford holomorphic. These conditions do not in general imply that the differential of f is given in terms of Clifford multiplication, which turns out to be a very restrictive condition.

If a complex or Clifford holormorphic function is twice continuously differentiable, then it is also harmonic, in the sense that its components are harmonic, as one can see from the identities relating the Laplacian to the various first-order differential operators mentioned before. We shall see in the next subsection that the condition of twice continuous-differentiability actually holds automatically for complex and Clifford holomorphic functions. Note that when $n = 1$, a continuously-differentiable function on an open interval (a, b) is "Clifford holomorphic" if and only if its derivative is equal to 0 on the interval, so that the function is constant, and a twice continuously-differentiable function on (a, b) is harmonic if and only if its second derivative is equal to 0 on the interval, which is to say that the function is of the form $\alpha\, x + \beta$.

If $h(z)$ is a twice continuously-differentiable function on a nonempty open subset U of $\mathbf{C}$ which is harmonic, then

$$f = \frac{\partial}{\partial z} h \tag{2.22}$$

is a continuously-differentiable function on U which is holomorphic. If $h(x)$ is a twice continuously-differentiable $\mathcal{C}(n)$-valued function on a nonempty open subset U of $\mathbf{R}^n$ which is harmonic, then the functions

$$f_L = \mathcal{D}_L h \quad \text{and} \quad f_R = \mathcal{D}_R h \tag{2.23}$$

are continuously-differentiable $\mathcal{C}(n)$-valued functions on U which are left and right Clifford holomorphic, respectively. If $h(x)$ happens to be real-valued, then $f_L = f_R$.

2.2 Weak derivatives, 1

Fix a positive integer n, and let U be a nonempty open subset of $\mathbf{R}^n$. Also let ϕ be a real, complex, or $\mathcal{C}(n)$-valued function on U. We say that ϕ has *restricted*

support in U if the set of $x \in U$ such that $\phi(x) \neq 0$ is bounded, and if there is a positive real number η such that

$$|x - y| \geq \eta \tag{2.24}$$

when $x \in U$, $y \in \mathbf{R}^n$, and $\phi(x) \neq 0$.

The sum of two functions with restricted support in U has rerstricted support in U, and the product of two functions on U, where at least one of the two functions has restricted support in U, also has restricted support in U. We shall mostly be concerned with continuous functions on U which have restricted support, and which then have nice properties in general, such as being bounded, because of basic results from advanced calculus. Notice that if $\phi(x)$ is a continuous function on U with restricted support, and if we extend $\phi(x)$ to all of $\mathbf{R}^n$ by setting $\phi(x) = 0$ when $x \in \mathbf{R}^n \backslash U$, then this extension is a continuous function on $\mathbf{R}^n$ with restricted support in $\mathbf{R}^n$.

Suppose that $\phi(x)$ be a continuous function on U with restricted support. The usual Riemann integral of $\phi(x)$ on U is then defined, which we can denote

$$\int_U \phi(x)\,dx. \tag{2.25}$$

As in the previous paragraph, we can just as well think of $\phi(x)$ as a continuous function on $\mathbf{R}^n$ with restricted support, and this integral as an integral over $\mathbf{R}^n$, even if only a bounded region in $\mathbf{R}^n$ is really involved in the integral.

Let $h(x)$ be a continuous function on U. We say that h is *weakly harmonic* if

$$\int_U (\Delta\phi(x))\,h(x)\,dx = 0 \tag{2.26}$$

for all twice continuously-differentiable functions ϕ on U with restricted support in U. Notice that if h is twice continuously-differentiable and harmonic in the usual sense, then h is harmonic in the weak sense, by integration by parts.

Let us consider for a moment the special case of a function $\phi(x)$ on $\mathbf{R}^n$ which is of the form $\rho(|x|^2)$, where $\rho(t)$ is a twice continuously-differentiable function on the real line. By the chain rule,

$$\Delta\phi(x) = 4\,|x|^2\,\rho''(|x|^2) + 2\,n\,\rho'(|x|^2). \tag{2.27}$$

In other words, $\Delta\phi$ is of the form $\sigma(|x|^2)$, where

$$\sigma(t) = 4\,t\,\rho''(t) + 2\,n\,\rho'(t). \tag{2.28}$$

Of course we are interested in functions with restricted support, and so let us assume that there is a positive real number r such that $\rho(t) = 0$ when $t \geq r$. For simplicity let us assume also that there is a positive real number ϵ such that $\rho(t)$ is constant for $t \leq \epsilon$. In particular, $\rho'(t)$, $\rho''(t)$, and $\sigma(t)$ are then equal to 0 for $t \leq \epsilon$.

By integration by parts, we automatically have

$$\int_{\mathbf{R}^n} \Delta\phi(x)\,dx = 0, \tag{2.29}$$

which is equivalent to

$$\int_0^\infty \sigma(t^2)\, t^{n-1}\, dt = 0, \tag{2.30}$$

and to

$$\int_0^\infty \sigma(t)\, t^{(n/2)-1}\, dt = 0. \tag{2.31}$$

Define a function $\theta(u)$ on $\mathbf{R}$ by

$$\theta(u) = \frac{1}{4} u^{-n/2} \int_0^u \sigma(t)\, t^{(n/2)-1}\, dt. \tag{2.32}$$

Thus $\theta(u) = 0$ when $u \le \epsilon$, and also when $u \ge r$.

By construction,

$$4\, u\, \theta'(u) + 2\, n\, \theta(u) = \sigma(u). \tag{2.33}$$

This is the same equation that ρ' satisfies. It follows that $\theta(u) = \rho'(u)$ for all $u \in \mathbf{R}$.

Conversely, suppose that $\sigma(t)$ is a continuous function on the real line which satisifies the conditions mentioned above, i.e., $\sigma(t) = 0$ when $t \le \epsilon$ and when $t \ge r$, where ϵ, r are positive real numbers, and the same integral as before is equal to 0. Then we can define θ again using the previous formula, so that θ is a continuous function on $\mathbf{R}$ such that $\theta(u) = 0$ when $u \le \epsilon$ and when $t \ge r$ in particular. We can define ρ on the real line so that $\rho' = \theta$ and $\rho(u) = 0$ when $u \ge r$, and we also have that $\rho(u)$ is constant on the set of $u \in \mathbf{R}$ such that $u \le \epsilon$.

In other words, we can reverse the process and start with a continuous function σ on the real line satisfying the conditions in the previous paragraph, and obtain a twice continuously-differentiable function ρ on the real line such that $\rho(u)$ is constant when $u \le \epsilon$ and equal to 0 when $u \ge r$. From the function ρ on the real line we get a function $\phi(x) = \rho(|x|^2)$ on $\mathbf{R}^n$. We want to use these functions as test functions for the weak harmonicity property.

Actually, we would like to use translates of such a function ϕ. For each $p \in \mathbf{R}^n$, put

$$\phi_p(x) = \phi(x - p) = \rho(|x - p|^2). \tag{2.34}$$

Thus

$$\Delta \phi_p(x) = \sigma(|x - p|^2). \tag{2.35}$$

Let us digress a moment with a useful definition. If A is a nonempty subset of $\mathbf{R}^n$ and x is a point in $\mathbf{R}^n$, then the *distance from x to A* is denoted $\mathrm{dist}(x, A)$ and defined by

$$\mathrm{dist}(x, A) = \inf\{|x - a| : a \in A\}. \tag{2.36}$$

One can check that $\mathrm{dist}(x, A) = 0$ if and only if x lies in the closure of A, and in particular this holds if and only if $x \in A$ when A is a closed subset of $\mathbf{R}^n$.

Now suppose that p is an element of U, and that r is a positive real number which satisfies

$$r^2 < \mathrm{dist}(p, \mathbf{R}^n \backslash U) \tag{2.37}$$

if U is a proper subset of $\mathbf{R}^n$, and otherwise is arbitrary if $U = \mathbf{R}^n$. Suppose that $\sigma(t)$ is a continuous function on the real line such that $\sigma(t) = 0$ when $t \le \epsilon$ for some $\epsilon > 0$, $\sigma(t) = 0$ when $t \ge r$ for the choice of r being used now, and $\sigma(t)$ satisfies the integral 0 condition discussed before. Thus we get associated functions θ, ρ on the real line and $\phi_p(x) = \rho(|x - p|^2)$ on $\mathbf{R}^n$, and ϕ_p has restricted support in U because of the constraint on r.

Let us apply this to our weakly harmonic function $h(x)$ on U. Namely, we have that

$$\int_U (\Delta \phi_p(x))\, h(x)\, dx = 0, \tag{2.38}$$

which is to say that

$$\int_U \sigma(|x - p|^2)\, h(x)\, dx = 0. \tag{2.39}$$

We want to use this with interesting choices of σ.

We can reformulate this condition by saying that if $\psi(x)$ is a continuous radial function on $\mathbf{R}^n$, so that $\psi(x)$ is actually a function of $|x|$, a is a positive real number such that $\psi(x) = 0$ when $|x| \ge a$, p is a point in U such that

$$\mathrm{dist}(p, \mathbf{R}^n \backslash U) > a, \tag{2.40}$$

and ψ satisfies the integral condition

$$\int_{\mathbf{R}^n} \psi(x)\, dx = 0, \tag{2.41}$$

then

$$\int_U \psi(x - p)\, h(x)\, dx = 0. \tag{2.42}$$

In other words, $\psi(x)$ corresponds to $\sigma(|x|^2)$. We have dropped the condition that $\psi(x)$ be equal to 0 when $|x|$ is sufficiently small, because one can reduce to this case through an approximation argument.

Let us reformulate this again as follows. Suppose that $b_1(x)$, $b_2(x)$ are continuous radial functions on $\mathbf{R}^n$, a is a positive real number such that $b_i(x) = 0$ when $|x| \ge a$, $i = 1, 2$, p is an element of $\mathbf{R}^n$ such that the distance from p to $\mathbf{R}^n \backslash U$ is greater than a, and

$$\int_{\mathbf{R}^n} b_1(x)\, dx = \int_{\mathbf{R}^n} b_2(x)\, dx = 1. \tag{2.43}$$

Then

$$\int_U b_1(x - p)\, h(x)\, dx = \int_U b_2(x - p)\, h(x)\, dx, \tag{2.44}$$

which follows from the statement in the previous paragraph by taking $\psi(x) = b_1(x) - b_2(x)$.

This implies that if $b(x)$ is a continuous radial function on $\mathbf{R}^n$, a is a positive real number such that $b(x) = 0$ when $|x| \ge a$, p is an element of U whose distance to $\mathbf{R}^n \backslash U$ is greater than a, and

$$\int_{\mathbf{R}^n} b(x)\, dx = 1, \tag{2.45}$$

then

$$\int_U b(x-p)\, h(x)\, dx = h(p). \tag{2.46}$$

To get this from the previous statement, one can take $b_1(x) = b(x)$, and choose $b_2(x)$ so that it is concentrated as near to p as one wants. Because h is continuous, the integral of b_2 times h is approximately equal to $h(p)$, and in the limit we get the desired formula.

In fact, if p is a point in U and t is a positive real number such that

$$t < \mathrm{dist}(p, \mathbf{R}^n \backslash U), \tag{2.47}$$

then

$$\mathrm{average}_{\overline{B}(p,t)} h = h(p). \tag{2.48}$$

This is called the *mean value property* for the continuous function h on U. It is easy to derive this from the previous statement by approximating the average of h on a closed ball in U by integrals against continuous radial functions, and conversely one can derive the previous assertion for integrals of h against continuous radial functions from this one about averages on closed balls.

To summarize, a continuous function $h(x)$ on a nonempty open subset of $\mathbf{R}^n$ which is weakly harmonic also satisfies the aforementioned mean value property.

2.3 Weak derivatives, 2

Let U be a nonempty open subset of $\mathbf{R}^n$, and let h be a continuous function on U which satisfies the mean value property. Let $b(x)$ be a continuous radial function on $\mathbf{R}^n$ and r a positive real number such that $b(x) = 0$ when $|x| \ge r$ and

$$\int_{\mathbf{R}^n} b(x)\, dx = 1. \tag{2.49}$$

On the set

$$U_r = \{p \in U : \mathrm{dist}(p, \mathbf{R}^n \backslash U) > r\}, \tag{2.50}$$

which one can check is an open subset of U, we have that

$$h(p) = \int_U b(x-p)\, h(x)\, dx. \tag{2.51}$$

We may as well take $b(x)$ to be as smooth as we like here, and so we choose $b(x)$ so that it is continuously differentiable of all orders. By differentiating under the integral sign, it follows that $h(x)$ is in fact continuously differentiable

of all orders on U_r. This works for all $r > 0$, and therefore $h(x)$ is continuously differentiable of all orders on U.

Thus a continuous function which is weakly harmonic is also continuously differentiable of all orders, and twice continuously-differentiable in particular. Such a function is harmonic in the usual sense. Similarly, a continuous function which satisfies the mean value property is harmonic.

Recall that an open subset U of $\mathbf{R}^n$ is said to be *connected* if for every pair of points $x, y \in U$ there is a continuous path in U that goes from x to y; equivalently, U is not connected if it can be expressed as the union of two disjoint nonempty open subsets of $\mathbf{R}^n$. If h is a continuous real-valued function on a nonempty connected open subset U of $\mathbf{R}^n$ which satisfies the mean value property, and if p is a point in U such that

$$h(x) \le h(p) \tag{2.52}$$

for all $x \in U$, then one can check that

$$h(x) = h(p) \tag{2.53}$$

for all $x \in U$, because $\{x \in U : h(x) < h(p)\}$ is automatically an open subset of U since h is continuous, while $\{x \in U : h(x) = h(p)\}$ is a nonempty open subset of U under the present conditions. For the same reasons, if q is a point in U such that

$$h(x) \ge h(q) \tag{2.54}$$

for all $x \in U$, then

$$h(x) = h(q) \tag{2.55}$$

for all $x \in U$.

2.4 Weak derivatives, 3

Let U be a nonempty open subset of $\mathbf{C}$, and let $f(z)$ be a continuous complex-valued function on U. We say that $f(z)$ is *weakly holomorphic* if

$$\int_U \left(\frac{\partial}{\partial \overline{z}} \psi(z) \right) f(z)\, dz = 0 \tag{2.56}$$

for all continuously differentiable complex-valued functions $\psi(z)$ on U with restricted support in U. If $f(z)$ is continuously differentiable and holomorphic, then $f(z)$ is weakly holomorphic, by integration by parts.

Suppose that $f(z)$ is weakly holomorphic on U. We can choose the test function $\psi(z)$ above to be of the form

$$\psi(z) = \frac{\partial}{\partial z} \phi(z), \tag{2.57}$$

where $\phi(z)$ is a twice continuously differentiable function with restricted support in U, and it follows that

$$\int_U (\Delta \phi(z))\, f(z)\, dz = 0, \tag{2.58}$$

so that $f(z)$ is weakly harmonic. Thus we obtain that $f(z)$ is continuously differentiable of all orders on U, and that $f(z)$ is holomorphic in the usual sense.

Now let U be a nonempty open subset of $\mathbf{R}^n$, and let $f(x)$ be a continuous $\mathcal{C}(n)$-valued function on U. We say that $f(x)$ is *weakly left Clifford holomorphic* if

$$\int_U \mathcal{D}_R \psi_1(x)\, f(x)\, dx = 0 \tag{2.59}$$

for all continuously-differentiable $\mathcal{C}(n)$-valued functions ψ_1 with restricted support in U, and we say that $f(x)$ is *weakly right Clifford holomorphic* if

$$\int_U f(x)\, \mathcal{D}_L \psi_2(x)\, dx = 0 \tag{2.60}$$

for all continuously-differentiable $\mathcal{C}(n)$-valued functions ψ_2 with restricted support in U. Again one can choose ψ_1, ψ_2 to be of the form $\mathcal{D}_R\phi$, $\mathcal{D}_L\phi$ for a twice continuously-differentiable function ϕ with restricted support in U to obtain that a weakly left or right Clifford holomorphic function is weakly harmonic, and hence harmonic, and thus continuously differentiable of all orders and left or right Clifford holomorphic function in the usual sense.

3 Harmonic and holomorphic functions, 2

3.1 Poisson kernels, 1

Fix a positive integer n. By a *multi-index* we mean an n-tuple

$$\alpha = (\alpha_1, \dots, \alpha_n) \tag{3.1}$$

where each α_i is a nonnegative integer. In this case the *degree* of the multi-index α is denoted $d(\alpha)$ and defined by

$$d(\alpha) = \alpha_1 + \cdots + \alpha_n. \tag{3.2}$$

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index. The corresponding *monomial* is the function of $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ defined by

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \tag{3.3}$$

The *degree* of the monomial x^α is defined to be the degree of the multi-index α.

By a *polynomial* on $\mathbf{R}^n$ we mean a function which is a finite linear combination of monomials. A *harmonic polynomial* is simply a polynomial which is harmonic. For a polynomial $p(x)$ on $\mathbf{R}^n$, the Laplacian $\Delta p(x)$ can be expressed algeraically, and thus the property of being harmonic can also be expressed algebraically.

A polynomial $p(x)$ on $\mathbf{R}^n$ is said to be *homogeneous of degree* d, where d is a nonnegative integer, if $p(x)$ is a finite linear combination of monomials x^α with $d(\alpha) = d$. This is equivalent to saying that

$$p(t\,x) = t^d\,p(x) \tag{3.4}$$

for all $t \in \mathbf{R}$ and all $x \in \mathbf{R}^n$. Every polynomial on $\mathbf{R}^n$ can be decomposed into homogeneous parts in a simple way.

If $p(x)$ is a homogeneous polynomial on $\mathbf{R}^n$ of degree d, then the Lapacian $\Delta p(x)$ of $p(x)$ is equal to 0 if $d = 0, 1$ and is a homogeneous polynomial of degree $d-2$ when $d \ge 2$. If $h(x)$ is a harmonic polynomial on $\mathbf{R}^n$, then its homogeneous components are also harmonic. If $q(x)$ is a homogeneous polynomial of degree ℓ, then $|x|^2\,q(x)$ is a homogeneous polynomial of degree $\ell + 2$.

Suppose that $p(x)$ is a homogeneous polynomial on $\mathbf{R}^n$ of degree d. Then

$$\sum_{j=1}^n x_j\,\frac{\partial}{\partial x_j}\,p(x) = d\,p(x), \tag{3.5}$$

which is known as "Euler's identity". This follows by computing the derivative of $p(t\,x)$ in t and setting $t = 1$, where the derivative is computed using the chain rule, and by using homogeneity to write $p(t, x)$ as $t^d\,p(x)$ and differentiating t^d directly.

If $q(x)$ is a polynomial on $\mathbf{R}^n$, then

$$\Delta(|x|^2\,q(x)) = 2\,n\,q(x) + 4\sum_{j=1}^n x_j\,\frac{\partial}{\partial x_j}\,q(x) + |x|^2\,\Delta q(x). \tag{3.6}$$

Assuming also that $q(x)$ is homogeneous of degree ℓ, it follows that

$$\Delta(|x|^2\,q(x)) = 2\,(n + 2\,\ell)\,q(x) + |x|^2\,\Delta q(x). \tag{3.7}$$

Supposing further that $q(x)$ is harmonic, we obtain that

$$\Delta(|x|^2\,q(x)) = 2\,(n + 2\,\ell)\,q(x), \tag{3.8}$$

and more generally $\Delta(|x|^{2k}\,q(x))$ is a positive integer multiple of $|x|^{2k-2}\,q(x)$ when k is a positive integer.

A key result now is that if $p(x)$ is any polynomial on $\mathbf{R}^n$, then $p(x)$ can be written as a finite linear combination of polynomials of the form $|x|^{2k}\,h(x)$, where k is a nonnegative integer and $h(x)$ is a harmonic polynomial. To be more precise, if $p(x)$ is a homogeneous polynomial of degree d, then $p(x)$ can be expressed as a finite linear combination of polynomials of the form $|x|^{2k}\,h(x)$, where k is a nonnegative integer and $h(x)$ is a homogeneous harmonic polynomial of degree ℓ. This can be verified using the computations above.

3.2 Poisson kernels, 2

Fix a positive integer n, and let $\mathbf{S}^{n-1}$ denote the unit sphere in $\mathbf{R}^n$, so that

$$\mathbf{S}^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}. \tag{3.9}$$

Also let $f(x)$ be a continuous real-valued function on $\mathbf{S}^{n-1}$. We are interested in continuous real-valued functions $h(x)$ on the closed unit ball $\overline{B}(0,1)$ which agree with $f(x)$ on the boundary $\mathbf{S}^{n-1}$ and which are harmonic on the open unit ball $B(0,1)$.

Suppose that $h(x)$ is a continuous real-valued function on $\overline{B}(0,1)$ which is harmonic on $B(0,1)$. By standard results from advanced calculus, $h(x)$ attains its maximum and minimum on $\overline{B}(0,1)$, and we have seen that if the maximum or minimum are attained in the interior $B(0,1)$, then $h(x)$ is constant. As a consequence, the maximum and minimum are always attained on the boundary $\mathbf{S}^{n-1}$, and it follows that

$$\sup\{|h(x)| : x \in \overline{B}(0,1)\} = \sup\{|h(x)| : x \in \mathbf{S}^{n-1}\}, \tag{3.10}$$

and in particular that $h(x) = 0$ for all $x \in \overline{B}(0,1)$ when $h(x) = 0$ for all $x \in \mathbf{S}^{n-1}$.

Thus a continuous real-valued function $h(x)$ on $\overline{B}(0,1)$ which is harmonic on $B(0,1)$ is uniquely determined by its boundary values, i.e., if $h_1(x)$, $h_2(x)$ are two continuous real-valued functions on $\overline{B}(0,1)$ which are harmonic on $B(0,1)$ and which are equal on $\mathbf{S}^{n-1}$, then $h_1(x) - h_2(x)$ is a continuous real-valued function on $\overline{B}(0,1)$ which is harmonic on $B(0,1)$ and equal to 0 on $\mathbf{S}^{n-1}$, and hence is equal to 0 on all of $\overline{B}(0,1)$. Let us now consider the question of existence of continuous functions on $\overline{B}(0,1)$ which are harmonic on $B(0,1)$ and whose values on the boundary $\mathbf{S}^{n-1}$ are prescribed in advance. When $n = 1$ the existence result is immediate, because $\mathbf{S}^{n-1}$ consists of the two points 1, -1, and affine functions are harmonic.

In general, if $f(x)$ is a continuous real-valued function on $\mathbf{S}^{n-1}$, then there is a sequence of polynomials $\{p_j(x)\}_{j=1}^\infty$ on $\mathbf{R}^n$ such that the restrictions of the $p_j(x)$'s to $\mathbf{S}^{n-1}$ converge uniformly to $f(x)$. This is a well-known result from advanced calculus. By the results of the previous subsection, there are harmonic polynomials $h_j(x)$ on $\mathbf{R}^n$ which agree with the $p_j(x)$'s on $\mathbf{S}^{n-1}$.

For positive integers j, k we have that

$$\begin{aligned}
&\sup\{|h_j(x) - h_k(x)| : x \in \overline{B}(0,1)\} \\
&\qquad = \sup\{|h_j(x) - h_k(x)| : x \in \mathbf{S}^{n-1}\} \\
&\qquad = \sup\{|p_j(x) - p_k(x)| : x \in \mathbf{S}^{n-1}\}.
\end{aligned} \tag{3.11}$$

Using this one can check that $\{h_j(x)\}_{j=1}^\infty$ converges uniformly to a continuous function $h(x)$ on $\overline{B}(0,1)$ which agrees with $f(x)$ on the boundary $\mathbf{S}^{n-1}$. Also, $h(x)$ is harmonic on $B(0,1)$, because it follows from uniform convergence that $h(x)$ is weakly harmonic.

One can also look for an integral formula for a harmonic extension on the closed unit ball of a continuous function on the unit sphere. Namely, if $f(x)$ is a continuous function on $\mathbf{S}^{n-1}$, one would like to write

$$h(x) = \int_{\mathbf{S}^{n-1}} P(x,y)\, f(y)\, dy \tag{3.12}$$

for a harmonic function $h(x)$ on $B(0,1)$ with boundary values $f(x)$. Here dy denotes the usual element of surface integration in $\mathbf{R}^n$.

More precisely, one would like $P(x,y)$ to be a continuous function defined for $x \in B(0,1)$ and $y \in \mathbf{S}^{n-1}$. In order for $h(x)$ to be harmonic, $P(x,y)$ should be harmonic as a function of $x \in B(0,1)$ for each $y \in \mathbf{S}^{n-1}$. In order for the boundary values of h to be given by f on $\mathbf{S}^{n-1}$, for each $z \in \mathbf{S}^{n-1}$ the function $P(x,y)$, as a function of y on $\mathbf{S}^{n-1}$, should converge to the Dirac delta function $\delta_z(y)$ at z as $x \in B(0,1)$ tends to z, in a suitable sense, which means in particular that $P(x,y)$ should tend to 0 for $y \neq z$ as x tends to z.

Because $h(x)$ should be equal to 1 for all $x \in B(0,1)$ when $f(y) = 1$ for all $y \in \mathbf{S}^{n-1}$, we should have

$$\int_{\mathbf{S}^{n-1}} P(x,y)\, dy = 1 \tag{3.13}$$

for all $x \in B(0,1)$. If $f(y)$ is a nonnegative real-valued function on $\mathbf{S}^{n-1}$, then $h(x)$ should be a nonnegative real-valued function on $B(0,1)$, and this leads to the condition that $P(x,y)$ should be a nonnegative real number for all $x \in B(0,1)$ and $y \in \mathbf{S}^{n-1}$. Also, $P(0,y)$ should be equal to $1/\nu_{n-1}$ for all $y \in \mathbf{S}^{n-1}$, where ν_{n-1} denotes the area of the unit sphere $\mathbf{S}^{n-1}$.

In fact there is a function $P(x,y)$ with these properties, called the *Poisson kernel for the unit ball*. Specifically, $P(x,y)$ is given by the formula

$$P(x,y) = \frac{1}{\nu_{n-1}} \frac{1 - |x|^2}{|x-y|^n}. \tag{3.14}$$

The Poisson kernel is unique, and it provides another approach to the existence of continuous functions on the closed unit ball which are harmonic on the open unit ball and have prescribed boundary values.

Similarly, on any closed ball $\overline{B}(w,r)$ a continuous function on the boundary sphere can be extended to a unique continuous function on the closed ball which is harmonic on the corresponding open ball, and the extension on the open ball can be expressed by an analogous Poisson integral of the boundary values. Let us note that the Poisson kernel is real-analytic as a function of x, which is to say that it admits local representations in terms of power series. Each harmonic function on a nonempty subset of $\mathbf{R}^n$ is locally given as a Poisson integral, and as a result it follows that each harmonic function is also real analytic.

Now let us describe an application of the existence of harmonic functions on balls in $\mathbf{R}^n$ with prescribed continuous boundary values. Let U be a nonempty open subset of $\mathbf{R}^n$, and let $h(x)$ be a continuous function on U. We say that

$h(x)$ satisfies the *weak mean value property* if for each $x \in U$ there is a positive real number $r(x)$ such that the usual mean value property holds for h for balls centered at x and with radius less than $r(x)$, which is to say that

$$\text{average}_{\overline{B}(x,r)} h = h(x) \tag{3.15}$$

for all positive real numbers r such that $r < r(x)$.

Suppose that U is a nonempty connected open subset of $\mathbf{R}^n$, and that $h(x)$ is a continuous real-valued function on U which satisfies the weak mean value function. If p is a point in U at which h attains its maximum, so that

$$h(x) \le h(p) \tag{3.16}$$

for all $x \in U$, then one can show that h is constant on U, with $h(x) = h(p)$ for all $x \in U$. Namely, $\{x \in U : h(x) < h(p)\}$ is an open subset of U simply because h is continuous, while $\{x \in U : h(x) = h(p)\}$ is an open subset of U because of the weak mean value property and the assumption that h attains its maximum at p, and hence $\{x \in U : h(x) = h(p)\} = U$ by the connectedness of U.

Another feature of the weak mean value property is that it is preserved by restricting a function to a smaller open set. As a result of this and the maximum principle in the preceding paragraph, it follows that a function which satisfies the weak mean value property on a nonempty open subset of $\mathbf{R}^n$ can be represented locally as a harmonic function on a ball, namely a harmonic function whose boundary values are the given function. In short, a continuous function on a nonempty open subset of $\mathbf{R}^n$ which satisfies the weak mean value property is actually harmonic.

3.3 Poisson kernels, 3

Fix a positive integer n again, and let us identify $\mathbf{R}^{n+1}$ with the Cartesian product $\mathbf{R}^n \times \mathbf{R}$, and write (x,t), $x \in \mathbf{R}^n$, $t \in \mathbf{R}$, for an element of $\mathbf{R}^n$. The *upper half-space* in $\mathbf{R}^{n+1}$ is denoted $\mathcal{U}^{n+1}$ and defined by

$$\mathcal{U}^{n+1} = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} : t > 0\}, \tag{3.17}$$

and the corresponding closed half-space is given by

$$\overline{\mathcal{U}}^{n+1} = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} : t \ge 0\}. \tag{3.18}$$

We shall be interested in continuous functions on the closed half-space which are harmonic on the open half-space.

Let us begin with a version of the *Schwarz reflection principle*. Namely, if $h(x,t)$ is a continuous function on the closed upper half-space which is harmonic on the open upper half-space, and if

$$h(x,0) = 0 \tag{3.19}$$

for all $x \in \mathbf{R}^n$, then we can extend $h(x,t)$ to all of $\mathbf{R}^n \times \mathbf{R}$ by setting

$$h(x,-t) = -h(x,t) \tag{3.20}$$

when $t > 0$, and this extension is harmonic. Indeed, it is easy to see that this extension satisfies the weak mean value property on $\mathbf{R}^n \times \mathbf{R}$.

Next, suppose that $f(x)$ is a harmonic function on $\mathbf{R}^m$ which is *bounded*, so that there is a positive real number C so that

$$|f(x)| \le C \tag{3.21}$$

for all $x \in \mathbf{R}^m$. In this case one can show that $f(x)$ is a constant function. Indeed, for any $x, y \in \mathbf{R}^m$ and any positive real number r, we have that

$$f(x) - f(y) = \mathrm{average}_{\overline{B}(x,r)} f - \mathrm{average}_{\overline{B}(x,r)} f, \tag{3.22}$$

and one can show that the right side tends to 0 as $r \to \infty$ because f is bounded.

In fact, there is a generalization of this for harmonic functions $f(x)$ on $\mathbf{R}^m$ which are of at most polynomial growth, in the sense that there is a positive real number C and a positive integer ℓ such that

$$|f(x)| \le C\,(1 + |x|)^\ell \tag{3.23}$$

for all $x \in \mathbf{R}^m$. In this case one can show that $f(x)$ is a polynomial of degree less than or equal to ℓ. For instance, one can show that the ℓth order derivatives of $f(x)$ are bounded harmonic functions on $\mathbf{R}^m$, and hence are constant.

At any rate, we conclude that if $h_1(x,t)$, $h_2(x,t)$ are bounded continuous functions on the closed upper half-space in $\mathbf{R}^m \times \mathbf{R}$ which are harmonic on the open half-space and which are equal on the boundary, then they are equal on the whole closed half-space. Indeed, under these conditions $h_1(x,t) - h_2(x,t)$ is a bounded continuous function on the closed half-space which is harmonic on the open half-space and equal to 0 when $t = 0$, and one can use reflection to get a bounded harmonic function on all of $\mathbf{R}^n \times \mathbf{R}$, which is therefore constant, and hence equal to 0. Notice that $h(x,t) = t$ is a harmonic function on $\mathbf{R}^n \times \mathbf{R}$ which is equal to 0 when $t = 0$ and which is not equal to 0 when $t \ne 0$.

Now let us consider the corresponding question of existence of bounded continuous functions on the closed half-space which are harmonic on the open half-space and which have prescribed boundary values.

We can begin with the point of view of Fourier analysis. Namely, suppose that ξ is an element of $\mathbf{R}^n$, and consider the bounded continuous function

$$\exp(i\xi \cdot x) \tag{3.24}$$

on $\mathbf{R}^n$, as a function of x. One can check that

$$\exp(-|\xi| t)\exp(i\xi \cdot x) \tag{3.25}$$

is a harmonic function of (x,t) on $\mathbf{R}^n \times \mathbf{R}$ which has modulus less than or equal to 1 when $t \ge 0$ and which agrees with the specified function of x when $t = 0$.

As in the case of the unit ball, there is a nice integral formula in this case. Namely, if $f(x)$ is a bounded continuous function on $\mathbf{R}^n$, then

$$h(x,t) = \int_{\mathbf{R}^n} P(x-y,t)\, f(y)\, dy \tag{3.26}$$

defines a bounded harmonic function on the upper-half space whose boundary values are equal to $f(x)$, where $P(x,t)$ is the *Poisson kernel* for the upper half-space, which is the function on the upper half-space given by

$$a_n \, \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}, \tag{3.27}$$

with a_n a positive real number.

To be more precise, if $f(x)$ is a bounded continuous function on $\mathbf{R}^n$, and we define $h(x,t)$ on the upper half-space using the formula above, and set $h(x,0) = f(x)$, then $h(x,t)$ is a bounded continuous function on the closed half-space which is harmonic on the open half-space. Harmonicity of $h(x,t)$ is a consequence of harmonicity of $P(x,t)$, as a function on the upper half-space. As a function of x, $P(x,t)$ tends to the Direc delta function at 0 as $t \to 0$ in a suitable sense, and this corresponds to the fact that $h(w,t)$ tends to $f(w)$ as $t \to 0$.

As in the case of the unit ball, $P(x,t)$ is positive everywhere, and this corresponds to the fact that $h(x,t)$ is positive everywhere when $f(x)$ is nonnegative and not identically equal to 0. Also,

$$\int_{\mathbf{R}^n} P(x,t)\, dx = 1 \tag{3.28}$$

for all $t > 0$, which corresponds to the fact that $h(x,t) = 1$ for all (x,t) in the upper half-space when $f(x) = 1$ for all $x \in \mathbf{R}^n$. The Poisson kernel $P(x,t)$ is the unique function on the upper half-space which represents bounded harmonic functions on the upper half-space that are continuous up to the boundary in this manner.

3.4 Fundamental solutions

Fix a positive integer n. Let us first consider the question of finding a *Newtonian kernel* $N_n(x)$ in dimension n, which should be a function on $\mathbf{R}^n$ whose Laplacian is equal to the Direc delta function at 0. This is to be interpreted in a weak sense, which is to say that

$$\int_{\mathbf{R}^n} N_n(x)\, \Delta\phi(x)\, dx = \phi(0) \tag{3.29}$$

when $\phi(x)$ is a twice continuously-differentiable function on $\mathbf{R}^n$ with restricted support.

On $\mathbf{R}^n \backslash \{0\}$, $N_n(x)$ should be a harmonic function. This does not mean that $N_n(x)$ is necessarily continuous at 0, however. It turns out that the singularity of

$N_n(x)$ at the origin is mild enough so that the above integral makes sense as an principal value integral, and behaves reasonably well, with absolute convergence in particular.

The Newtonian kernel is not uniquely determined by the condition that its Laplacian be equal to the Dirac delta function at 0 in the weak sense, because one could add to $N_n(x)$ any harmonic function on $\mathbf{R}^n$ and get a function with the same property. However, if one imposes the condition that $N_n(x)$ be a radial function, then it is unique except for adding a constant. This is because radial harmonic functions on $\mathbf{R}^n \backslash \{0\}$ are characterized by a homogeneous second-order ordinary differential equation, with a two-dimensional space of solutions, and constant functions account for a one-dimensional space of solutions.

When $n = 1$, one can take

$$N_1(x) = \frac{1}{2}\,|x|. \tag{3.30}$$

When $n = 2$ one can take

$$N_2(x) = b_2 \log |x|, \tag{3.31}$$

and when $n \geq 3$ one can take

$$N_n(x) = -b_n |x|^{2-n}, \tag{3.32}$$

where b_n, $n \geq 2$, are positive real numbers. This is not too difficult to check.

If y is a point in $\mathbf{R}^n$, then $N_n(x - y)$ has Laplacian, as a function of x, equal to the Dirac delta function at y in the weak sense, so that

$$\int_{\mathbf{R}^n} N_n(x - y)\, \Delta\phi(x)\, dx = \phi(y) \tag{3.33}$$

for all twice continuously-differentiable functions ϕ on $\mathbf{R}^n$ with restricted support. Another way to look at this is that if ψ is a continuous function on $\mathbf{R}^n$ with restricted support, then

$$\Psi(x) = \int_{\mathbf{R}^n} N_n(x - y)\, \psi(y)\, dy \tag{3.34}$$

is a continuous function on $\mathbf{R}^n$ such that

$$\Delta\Psi = \psi \tag{3.35}$$

in the weak sense, which is to say that

$$\int_{\mathbf{R}^n} \Psi(x)\, \Delta\phi(x)\, dx = \int_{\mathbf{R}^n} \psi(x)\, \phi(x)\, dx \tag{3.36}$$

for all twice continuously-differentiable functions ϕ on $\mathbf{R}^n$ with restriced support. In general, Ψ is not quite twice continuously-differentiable when ψ is continuous, but this is almost true, and Ψ is twice continuously-differentiable under additional mild conditions on ψ.

Of course the idea of a fundamental solution like this makes sense for differential operators in general.

For instance, when $n = 1$, one can consider the derivative of $N_1(x)$, which is equal to $-1/2$ when $x < 0$ and to $1/2$ when $x > 0$. One can make the convention of setting this function to be 0 at $x = 0$. At any rate, $N_1'(x)$ is a fundamental solution for d/dx in the sense that its derivative is equal to the Dirac delta function at 0 in the weak sense, so that

$$-\int_{\mathbf{R}} N_1'(x)\,\phi'(x)\,dx = \phi(0) \tag{3.37}$$

for all continuously-differentiable functions on the real line with restricted support.

Similarly, a nonzero multiple of $1/z$ defines a fundamental solution for the differential operator $\partial/\partial\overline{z}$. In other words, the $\partial/\partial\overline{z}$ derivative of $1/z$ is equal to a nonzero constant times the Dirac delta function at 0 on $\mathbf{C}$, in the sense that

$$\int_{\mathbf{C}} \frac{1}{z}\,\frac{\partial}{\partial\overline{z}}\phi(z)\,dz \tag{3.38}$$

is equal to a nonzero constant times $\phi(0)$ for all continuously-differentiable functions ϕ on $\mathbf{C}$ with restricted support. Note that $1/z$ is indeed holomorphic on $\mathbf{C}\backslash\{0\}$, as it should be to have $\partial/\partial\overline{z}$ derivative equal to 0 there, and that $1/z$ is a constant multiple of $\partial/\partial z N_2(z)$, identifying $\mathbf{R}^2$ with $\mathbf{C}$ in the usual manner.

For general n, the Clifford-valued function

$$E_n(x) = \frac{\sum_{j=1}^n x_i\,e_i}{|x|^n} \tag{3.39}$$

is a nonzero constant multiple of a fundamental solution for both $\mathcal{D}_L$ and $\mathcal{D}_R$. This means that

$$\int_{\mathbf{R}^n} \mathcal{D}_R\phi(x)\,E_n(x)\,dx \tag{3.40}$$

and

$$\int_{\mathbf{R}^n} E_n(x)\,\mathcal{D}_L\phi(x)\,dx \tag{3.41}$$

are equal to a constant multiple of $\phi(0)$ whenever $\phi(x)$ is a continuously-differentiable Clifford-valued function on $\mathbf{R}^n$ with restricted support. This function $E_n(x)$ is a constant multiple of $\mathcal{D}_L N_n(x) = \mathcal{D}_R N_n(x)$, and is both left and right Clifford holomorphic on $\mathbf{R}^n\backslash\{0\}$, as it should be.

4 Harmonic and holomorphic functions, 3

4.1 Subharmonic functions, 1

In dimension 1, the notion of subharmonicity reduces to that of convexity. Let (a, b) be an open interval in the real line, which is to say the set of real numbers x such that

$$a < x < b, \tag{4.1}$$

where of course we assume that $a < b$, and also we allow a to be $-\infty$ or a finite real number, and b to be a finite real number or $+\infty$, so that our interval may be unbounded, like the set of positive real numbers, or the whole real line itself. A real-valued function $f(x)$ is said to be convex if

$$f(\lambda\, x + (1-\lambda)\, y) \le \lambda\, f(x) + (1-\lambda)\, f(y) \tag{4.2}$$

for all $x, y \in (a,b)$ and all $\lambda \in (0,1)$.

It is a well-known exercise that convex functions are continuous. In the other direction, if one assumes that $f(x)$ is a continuous real-valued function on (a,b), then more restrictive convexity conditions imply the general one, e.g., it is enough to take $\lambda = 1/2$. If $f(x)$ is a twice differentiable function on (a,b), then $f(x)$ is convex if and only if $f''(x) \ge 0$ for all $x \in (a,b)$.

For any continuous real-valued function $f(x)$ on (a,b), we can say that $f'' \ge 0$ on (a,b) in the weak sense if

$$\int_a^b f(x)\, \phi''(x)\, dx \ge 0 \tag{4.3}$$

for all twice continuously-differentiable functions $\phi(x)$ with restricted support in (a,b) such that $\phi(x) \ge 0$ for all $x \in (a,b)$. If $f(x)$ is also twice continuously-differentiable, then $f'' \ge 0$ on (a,b) in the weak sense if and only if $f''(x) \ge 0$ for all $x \in (a,b)$, basically by integrating by parts twice. For any continuous real-valued function $f(x)$ on (a,b), one can show that $f(x)$ is convex if and only if $f'' \ge 0$ on (a,b) in the weak sense.

4.2 Subharmonic functions, 2

Let U be a nonempty open subset of $\mathbf{R}^n$, and let $f(x)$ be a real-valued continuous function on U. We say that $f(x)$ satisifies the weak sub-mean value property on U if for each $x \in U$ there is a positive real number $r(x)$ such that

$$\mathrm{average}_{\overline{B}(x,r)} f \ge f(x) \tag{4.4}$$

for all $r \in (0, r(x))$. To be more precise, this is a sub-mean value property for averages over balls, and we shall also consider versions with averages over spheres.

By definition, if a continuous real-valued function satisfies the weak sub-mean value property on an open set, it also satisfies this property when restricted to any open subset of this set. If U is a nonempty connected open subset of $\mathbf{R}^n$, $f(x)$ is a real-valued continuous function on U which satisfies the weak sub-mean value property, and p is an element of U at which $f(x)$ attains its maximum, so that

$$f(x) \le f(p), \tag{4.5}$$

then $f(x)$ is constant on U. As usual, this is because $\{x \in U : f(x) = f(p)\}$ is then an open subset of U, by the weak sub-mean value property, while $\{x \in U : f(x) < f(p)\}$ is an open subset of U by continuity.

Let U be a nonempty open subset of $\mathbf{R}^n$, let $f(x)$ be a real-valued continuous function on U, let z be an element of U, and let r be a positive real number such that

$$r < \operatorname{dist}(z, \mathbf{R}^n \backslash U). \tag{4.6}$$

Let $h(x)$ be the continuous real-valued function on $\overline{B}(z,r)$ which is equal to $f(x)$ when $|x-z| = r$ and which is harmonic on $B(z,r)$. Thus $f(x) - h(x)$ satisfies the weak sub-mean value property on $B(z,r)$, and since $f(x) - h(x)$ is equal to 0 on the boundary, it follows that $f(x) - h(x) \le 0$ on $B(z,r)$, which is to say that

$$f(x) \le h(x) \tag{4.7}$$

when $|x-z| < r$.

Because $h(x)$ is harmonic on $B(z,r)$, we have that

$$\int_{B(z,r)} h(x)\, b(x)\, dx = h(z) \tag{4.8}$$

whenever $b(x)$ is a continuous real-valued function with restricted support in $B(z,r)$ which is radial around z, in the sense that it depends only on $|x-z|$, and which satisfies

$$\int_{B(z,r)} b(x)\, dx = 1. \tag{4.9}$$

As a result,

$$\operatorname{average}_{\partial B(z,t)} h = h(z) \tag{4.10}$$

when $0 < t < r$, where $\partial B(z,t) = \{x \in \mathbf{R}^n : |x-z| = t\}$. This equation also holds when $t = r$, since $h(x)$ is continuous on $\overline{B}(z,r)$.

It follows that

$$\operatorname{average}_{\partial B(z,r)} f \ge \operatorname{average}_{\partial B(z,t)} f \tag{4.11}$$

when $0 < t < r$, and that

$$\operatorname{average}_{\partial B(z,r)} f \ge f(z). \tag{4.12}$$

This works for all radii $r < \operatorname{dist}(z, \mathbf{R}^n \backslash U)$, and by integrating over r one gets that if $\beta(x)$ is a nonnegative real-valued continuous function with restricted support in U which is radial about z, so that $\beta(x)$ only depends on $|x-z|$, and which satisfies

$$\int_U \beta(x)\, dx = 1, \tag{4.13}$$

then

$$\int_U f(x)\, \beta(x)\, dx \ge f(z). \tag{4.14}$$

Also,

$$\operatorname{average}_{\overline{B}(z,r)} f \ge f(z) \tag{4.15}$$

when $z \in U$ and $0 < r < \operatorname{dist}(z, \mathbf{R}^n \backslash U)$.

4.3 Subharmonic functions, 3

Let U be a nonempty open subset of $\mathbf{R}^n$. A real-valued continuous function $f(x)$ is said to be *subharmonic* if it satisfies the weak sub-mean value property on U, which is then equivalent to seemingly-stronger conditions, as in the previous subsection. If $f(x)$ is twice continuously-differentiable on U, then $\Delta f(x) \geq 0$ for all $x \in U$ in this case.

Conversely, suppose that $f(x)$ is a real-valued function on a nonempty open subset U of $\mathbf{R}^n$ such that f is twice continuously-differentiable and $\Delta f(x) \geq 0$. Actually, we can weaken this a bit by assuming that $f(x)$ is a real-valued continuous function on U such that

$$\liminf_{r \to 0} r^{-2} \left(\text{average}_{\overline{B}(x,r)} f - f(x) \right) \geq 0 \tag{4.16}$$

for all $x \in U$, or even

$$\limsup_{r \to 0} r^{-2} \left(\text{average}_{\overline{B}(x,r)} f - f(x) \right) \geq 0 \tag{4.17}$$

for all $x \in U$. In this case one can check that

$$f_\epsilon(x) = f(x) + \epsilon \, |x|^2 \tag{4.18}$$

satisfies the weak sub-mean value property on U for all $\epsilon > 0$. Hence

$$\text{average}_{\overline{B}(x,r)} f_\epsilon \geq f_\epsilon(x) \tag{4.19}$$

when $x \in U$ and $0 < r < \operatorname{dist}(x, \mathbf{R}^n \backslash U)$, and therefore

$$\text{average}_{\overline{B}(x,r)} f \geq f(x) \tag{4.20}$$

for the same x, r, so that $f(x)$ is a subharmonic function on U.

A real-valued continuous function $f(x)$ on a nonempty open subset U of $\mathbf{R}^n$ is said to be *subharmonic in the weak sense* if

$$\int_U f(x) \, \Delta\phi(x) \, dx \geq 0 \tag{4.21}$$

for all nonnegative real-valued twice continuously-differentiable functions $\phi(x)$ with restricted support in U. If $f(x)$ is twice continuously-differentiable, then it is easy to check that this is equivalent to $\Delta f(x) \geq 0$ for all $x \in U$, by integration by parts. It is not too difficult to extend this argument to show that a subharmonic function is subharmonic in the weak sense.

Conversely, a real-valued continuous function $f(x)$ on U which is subharmonic in the weak sense is subharmonic. This is analogous to the fact that a function which is harmonic in the weak sense satisfies the mean value property. As before, one can express differences of local radial averages of a function f as integrals of f times $\Delta\phi$ for twice continuously-differentiable functions with

restricted support in U, and now one should be a bit more careful to have $\phi \geq 0$ too.

Of course a real-valued continuous function $f(x)$ on an open subset U of $\mathbf{R}^n$ is harmonic if and only if $f(x)$ and $-f(x)$ are subharmonic. If f_1, f_2 are subharmonic functions on U and a_1, a_2 are nonnegative real numbers, then

$$a_1\, f_1 + a_2\, f_2 \tag{4.22}$$

is also a subharmonic function on U. Moreover,

$$\max(f_1, f_2) \tag{4.23}$$

is a subharmonic function on U in this case.

4.4 Holomorphic functions and subharmonicity

Let U be a nonempty open subset of $\mathbf{R}^n$, and let $f(x)$ be a real-valued continuous function on U. If $\alpha(t)$ is a real-valued convex function on $\mathbf{R}$, then

$$\alpha\Big(\mathrm{average}_{\overline{B}(x,r)} f\Big) \leq \mathrm{average}_{\overline{B}(x,r)} \alpha \circ f \tag{4.24}$$

for all $x \in U$ and $0 < r < \mathrm{dist}(x, \mathbf{R}^n \backslash U)$. In other words, the value of a convex function at an average of some numbers is less than or equal to the average of the values of the convex function at the same numbers.

As a consequence, if $f(x)$ is a real-valued harmonic function on U and $\alpha(t)$ is a convex function on the real line, then the composition $\alpha(f(x))$ is a real-valued continuous function on U which is subharmonic. Moreover, if $f(x)$ is a real-valued continuous subharmonic function on U and $\alpha(t)$ is a monotone increasing convex function on the real line, then the composition $\alpha(f(x))$ is a real-valued continuous subharmonic function on U. For instance, in the first case one can take $\alpha(t) = |t|^p$ and in the second case one can take $\alpha(t) = 0$ when $t < 0$, $\alpha(t) = t^p$ when $t \geq 0$, where p is a real number such that $p \geq 1$.

In fact, if U is a nonempty open subset of $\mathbf{C}$, and $f(z)$ is a complex-valued holomorphic function on U, then $|f(z)|^p$ is subharmonic for all $p > 0$. One also has that $\log |f(z)|$ is subharmonic, if one extends the notion of subharmonicity to allow for functions which take the value $-\infty$, or alternatively the real-valued function $\max(\log |f(z)|, \lambda)$ is subharmonic for every real number λ. Indeed, these assertions can be verified using the fact that $\log |f(z)|$ is harmonic on the set where $f(z) \neq 0$.

Now suppose that U is a nonempty open subset of $\mathbf{R}^n$, and that $f(x)$ is a $\mathcal{C}(n)$-valued function on U. We can define $|f(x)|$ by identifying $\mathcal{C}(n)$ with a Euclidean space, using the standard basis for the Clifford algebra. It turns out that if $f(x)$ is Clifford holomorphic, then $|f(x)|^p$ is subharmonic for a range of p's which includes some p's strictly less than 1. There are general results of this type for first-order constant coefficient systems of partial differential equations which are "elliptic" in the sense that their homogeneous solutions are harmonic, and there are various situations where this applies with various ranges of p. See [4, 5].

5 Miscellaneous

5.1 Linear fractional transformations

A *polynomial* on $\mathbf{C}$ is a function $P(z)$ of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \tag{5.1}$$

where n is a nonnegative integer and $a_0, \ldots, a_n$ are complex numbers. If $n \ge 1$ and $a_n \ne 0$, or if $n = 0$, then we say that $P(z)$ has degree n. If $P(z)$ is a polynomial of degree $n \ge 1$, then every complex number is attained as a value of $P(z)$ exactly n times if one counts multiplicities in an appropriate manner.

Let us write $\widehat{\mathbf{C}}$ for the *Riemann sphere*, which consists of the complex numbers $\mathbf{C}$ together with an extra "point at infinity" denoted ∞. This is topologically equivalent to the usual 2-dimensional sphere in the obvious manner. We also make the conventions that $1/\infty = 0$, $1/0 = \infty$, $a + \infty = \infty$ when $a \in \mathbf{C}$, $a \cdot \infty = \infty$ when $a \in \mathbf{C}$, $a \ne 0$, etc.

If $P(z)$ is a polynomial on $\mathbf{C}$, then P can be extended to a continuous mapping from the Riemann sphere in a natural way, namely, $P(\infty) = \infty$ when $P(z)$ has positive degree, and P takes the same value at ∞ as at other points when P is constant. More generally we can consider *rational functions* of the form

$$R(z) = \frac{P(z)}{Q(z)}, \tag{5.2}$$

where $Q(z)$ is a polynomial which is not identically equal to 0. To be more precise, $R(z)$ is then defined as a complex number when z is a complex number and $Q(z) \ne 0$, while if $Q(z) = 0$ or $z = \infty$ one can follow the usual conventions, so that the rational function remains continuous as a mapping from the Riemann sphere to itself.

Assuming that the polynomials $P(z)$, $Q(z)$ have no common factors, then the degree of the corresponding rational function $R(z)$ is defined to be the maximum of the degrees of $P(z)$ and $Q(z)$. If $R(z)$ is a rational function of degree $n \ge 1$, then each element of the Riemann sphere occurs as a value of $R(z)$ n times, with appropriate multiplicities. This is not too difficult to see.

A *linear fractional transformation* on the Riemann sphere is a rational function of degree 1. Explicitly, this is a mapping which can be expressed as

$$\frac{a\,z + b}{c\,z + d}, \tag{5.3}$$

where a, b, c, d are complex numbers such that

$$a\,d - b\,c \ne 0. \tag{5.4}$$

This condition ensures that the denominator in the linear fractional transformation is not identically equal to 0, and that the numerator and denominator are not simply constant multiples of each other. Notice that if we multiply a, b, c, and d by the same nonzero complex number, then this admissibility condition

is still satisfied, and that the corresponding linear fractional transformation on the Riemann sphere is in fact the same as the original one.

In fact, we can think of the complex numbers a, b, c, d as entries of a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{5.5}$$

whose determinant is then equal to $a\,c - b\,d$, so that the admissibility condition is equivalent to the requirement that this matrix be invertible. One can check that the composition of two linear fractional transformations is again a linear fractional transformations, and that this corresponds to multiplying the associated matrices in the usual way. In particular, the inverse of a linear fractional transformation can be given by the inverse of the associated matrix.

It is helpful to consider some special cases of linear fractional transformations, of the form

$$a\,z \tag{5.6}$$

for a nonzero complex number a,

$$z + b \tag{5.7}$$

for any complex number b, and

$$\frac{1}{z}. \tag{5.8}$$

One can check that every linear fractional transformation can be expressed as a combination of these basic types. Of course this can also be viewed in terms of 2×2 matrices.

Let us consider this in another way, which extends to higher dimensions. Namely, if n is a positive integer, let $\mathbf{CP}^n$ denote the space of complex lines through the origin in $\mathbf{C}^{n+1}$, the space of $(n + 1)$-tuples of complex numbers. In other words, we can start with $\mathbf{C}^{n+1}\backslash\{0\}$, and identify two elements z, w of this space when there is a nonzero complex number λ such that $w = \lambda\,z$.

If A is an invertible complex linear mapping on $\mathbf{C}^{n+1}$, then A takes the origin 0 in $\mathbf{C}^{n+1}$ to itself, and it takes complex lines in $\mathbf{C}^{n+1}$ to themselves, and thus induces a transformation on $\mathbf{CP}^n$. Multiplication of A by a nonzero complex number does not change the induced transformation on $\mathbf{CP}^n$. Compositions of invertible linear mappings on $\mathbf{C}^{n+1}$ correspond to compositions of the induced transformations on $\mathbf{CP}^n$, and in particular the inverse of an induced transformation on $\mathbf{CP}^n$ is induced by the inverse of the original linear mapping on $\mathbf{C}^{n+1}$.

There is a natural way to embedd $\mathbf{C}^n$ into $\mathbf{CP}^n$, which is to take an element z of $\mathbf{C}^n$ and extend it to a nonzero element of $\mathbf{C}^{n+1}$ by adding an $(n + 1)$th coordinate which is set equal to 1. In other words, we identify $\mathbf{C}^n$ with the affine hyperspace in $\mathbf{C}^{n+1}$ of points whose last coordinate is equal to 1, and then each point in this affine hyperspace determines a complex line in $\mathbf{C}^{n+1}$ through the origin, namely the complex line that passes through that point. This accounts for all the complex lines in $\mathbf{C}^{n+1}$ except for those which lie in the linear subspace of points whose last coordinate is equal to 0.

Roughly speaking then, $\mathbf{C}^n$ looks like $\mathbf{C}^n$ together with a copy of $\mathbf{CP}^{n-1}$, at least when $n \geq 2$. When $n = 1$, the embedding of $\mathbf{C}$ into $\mathbf{CP}^1$ accounts for all the lines in $\mathbf{C}^2$ through the origin except for one line, which is the set of points with second coordinate equal to 0. Thus $\mathbf{CP}^1$ looks like $\mathbf{C}$ with one additional element, which gives another way to think about the Riemann sphere.

Linear fractional transformations on the Riemann sphere are then the same as transformations on $\mathbf{CP}^1$ induced by invertible complex linear transformations on $\mathbf{C}^2$. To be more explicit, an invertible linear transformation on $\mathbf{C}^2$ can be given as

$$(z, w) \mapsto (a\,z + b\,w, c\,z + d\,w), \tag{5.9}$$

where again a, b, c, d are complex numbers such that $a\,c - b\,d \neq 0$. This leads to the earlier formula through the identifications described above.

There is another and rather different way to extend the idea of linear fractional transformations to higher dimensions. For this we think of $\mathbf{R}^2$ as being identified with $\mathbf{C}$ in the usual way. Now we work on $\mathbf{R}^n$, and on $\widehat{\mathbf{R}^n} \simeq \mathbf{R}^n \cup \{\infty\}$, which can be identified topologically with an n-dimensional sphere.

To describe this generalization, one can go back to the idea of building blocks. Namely, one can use translations, dilations, orthogonal transformations, and inversion about the unit sphere in $\mathbf{R}^n$ as the basic building blocks. One then gets a nice family of transformations on $\widehat{\mathbf{R}^n}$ generated by these.

Actually, this would correspond to also allowing complex conjugation as a transformation on the Riemann sphere. Alternatively, in n dimensions, one can add the requirement that the transformations preserve orientations. At any rate, this generalization is closely connected to conformal geometry in n dimensions.

5.2 Power series and the exponential function

Consider a power series

$$\sum_{n=0}^{\infty} a_n\, z^n, \tag{5.10}$$

where the a_n's are complex numbers and z is a complex variable. We shall be interested in the set of $z \in \mathbf{C}$ for which this series converges. Of course this series converges trivially when $z = 0$.

If the series converges for some $z_0 \in \mathbf{C}$, then we have that the sequence $\{a_n\, z_0^n\}_{n=0}^{\infty}$ tends to 0, and is bounded in particular, so that there is a nonnegative real number C such that

$$|a_n|\, |z_0|^n \leq C \tag{5.11}$$

for all n. Assume that $z_0 \neq 0$ and that z is a complex number such that

$$|z| < |z_0|. \tag{5.12}$$

Under these conditions, we have that

$$|a_n|\, |z|^n \leq C \left(\frac{|z|}{|z_0|}\right)^n, \tag{5.13}$$

and hence $\sum_{n=0}^{\infty} a_n \, z^n$ converges absolutely by comparison with a convergent geometric series.

Using this simple remark, one can check that either the power series converges only for $z = 0$, or there is a positive real number R such that the series converges absolutely when $|z| < R$ and does not converge when $|z| > R$, or the series converges absolutely for all $z \in \mathbf{C}$. In the first case we can set $R = 0$, and in the third case we can set $R = \infty$. We call R the radius of convergence of the power series.

If the power series converges absolutely for some complex number z_0, then it also converges absolutely for all complex numbers z such that $|z| \le |z_0|$. In general, if a power series has a radius of convergence R which is positive and finite, then the series might converge for no complex numbers z with $|z| = R$, or for some of them, or for all of them. For instance, the series

$$\sum_{n=0}^{\infty} z^n \tag{5.14}$$

has radius of convergence equal to 1 and converges for no $z \in \mathbf{C}$ with $|z| = 1$,

$$\sum_{n=1}^{\infty} \frac{z^n}{n} \tag{5.15}$$

has radius of convergence 1 and converges for $z \in \mathbf{C}$ with $|z| = 1$ and $z \ne 1$, and does not converge when $z = 1$, and

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} \tag{5.16}$$

has radius of convergence 1 and converges absolutely for all $z \in \mathbf{C}$ with $|z| \le 1$.

Let

$$\sum_{n=0}^{\infty} A_n, \quad \sum_{n=0}^{\infty} B_n \tag{5.17}$$

be two series of complex numbers, and define an associated series

$$\sum_{n=0}^{\infty} C_n \tag{5.18}$$

by

$$C_n = \sum_{j=0}^{n} A_j \, B_{n-j}. \tag{5.19}$$

This is called the Cauchy product of the original two series. Formally, the sum of the C_n's is equal to the product of the sums of the A_n's and the B_n's, and indeed this is the case if the A_n's and B_n's are equal to 0 for all but finitely many n. If the original series are power series, with $A_n = a_n \, z_n$ and $B_n = b_n \, z^n$, then $C_n = c_n \, z^n$, with $c_n = \sum_{j=0}^{n} a_j \, b_{n-j}$.

It is not too difficult to show that if $\sum A_n$, $\sum B_n$ converge absolutely, then the Cauchy product series $\sum C_n$ also converges absolutely, and the sum of the Cauchy product is equal to the products of the sums of the initial series. One can also show that if both series converge and at least one converges absolutely, then the Cauchy product converges, and the sum of the Cauchy product series is equal to the product of the sums of the original series. It is not true in general that the Cauchy product series converges when the two initial series converge.

There is a nice result though which says that if the two initial series converge and the Cauchy product series converges, then the sum of the Cauchy product series is equal to the product of the sums of the two initial series. To show this, one can use the notion of *Abel summability*. Basically, it is convenient to view the series as power series.

Specifically, for a given series $\sum_{n=0}^{\infty} A_n$, the associated Abel sums are defined by

$$\sum_{n=0}^{\infty} r^n A_n, \tag{5.20}$$

where r is a positive real number such that $r < 1$. We assume that these series converge for all $r \in (0,1)$, which is the same as saying that they converge absolutely for all $r \in (0,1)$, as before. This holds under modest assumptions on the size of the A_n's.

The series $\sum_{n=0}^{\infty} A_n$ is said to be Abel summable if the Abel sums exist for all $r \in (0,1)$, and if the limit of the Abel sums exist as $r \to 1-$. This limit is then called the Abel sum of the series $\sum_{n=0}^{\infty} A_n$. For instance, if the A_n's are nonnegative real numbers, then the Abel sums are monotone increasing as a function of r, and it is easy to see that the series is Abel summable if and only if it converges in the usual sense, and the Abel sum is equal to the ordinary sum.

If $\sum A_n$ converges absolutely, then it is again easy to see that $\sum A_n$ is Abel summable, and that the Abel sum is equal to the ordinary sum. It turns out that this works in general when $\sum A_n$ converges. One can approach this by expressing the Abel sums in terms of the partial sums of the original series, noting also that the Abel sums converge for all $r \in (0,1)$ because the A_n's are bounded.

There are also series which are Abel summable and which do not converge themselves. A basic family of examples is given by the series

$$\sum_{n=0}^{\infty} z^n, \tag{5.21}$$

where z is a complex number such that $|z| = 1$ and $z \neq 1$. When $r \in (0,1)$, we have that

$$\sum_{n=0}^{\infty} r^n z^n = \frac{1}{1 - r\,z}, \tag{5.22}$$

and this converges to $(1-z)^{-1}$ as $r \to 1-$.

Now suppose that $\sum A_n$, $\sum B_n$ are two convergent series of complex numbers. In particular their terms are bounded, and the Cauchy product series

$\sum C_n$ has the property that $C_n = O(n+1)$, say, which is sufficient to ensure that $\sum r^n C_n$ converges for all $r \in (0,1)$. If we assume that $\sum C_n$ converges, then the sum of this series is the same as the limit of the Abel sums $\sum r^n C_n$, and these Abel sums are equal to the product of the Abel sums $\sum r^n A_n$, $\sum r^n B_n$, since we have absolute convergence when $r \in (0,1)$, and it follows that $\sum C_n$ is equal to $\sum A_n$ times $\sum B_n$.

Suppose that we have a power series

$$\sum_{n=0}^{\infty} \alpha_n z^n \tag{5.23}$$

which converges for $z \in \mathbf{C}$ with $|z| < R$ for some $R > 0$. A basic result states that if $0 < r < R$, then the partial sums of this series converge uniformly to the whole sum for $z \in \mathbf{C}$ such that $|z| \le r$. In particular, it follows that the power series defines a continuous function on the disk

$$\{z \in \mathbf{C} : |z| < R\}. \tag{5.24}$$

Under these conditions we also have that the series

$$\sum_{n=0}^{\infty} n\, \alpha_n z^n \tag{5.25}$$

converges for $|z| < R$. The original series defines a holomorphic function $f(z)$ on the open disk in the complex plane with center 0 and radius R, and this series represents the complex derivative $f'(z)$ of $f(z)$.

Now let us look at a specific power series. Namely, consider the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}, \tag{5.26}$$

where as usual $n!$ denotes "n factorial", the product of the integer from 1 to n, which is interpreted as being equal to 1 when $n = 0$. For each positive integer k, we have that

$$\frac{1}{n!} \le \frac{1}{(k-1)!} \frac{1}{k^{n-k}} \tag{5.27}$$

when $n \ge k$, and one can use this to show that our power series converges for all z in $\mathbf{C}$.

Of course this is the series for the exponential function $\exp(z)$. With this definition one can see directly that

$$\exp(z+w) = \exp(z)\, exp(w) \tag{5.28}$$

for all $z, w \in \mathbf{C}$. This is an example of the general result for Cauchy products of absolutely convergent series, as one can check using the binomial theorem.

Clearly $\exp(0) = 1$. When z is a positive real number, $\exp(z)$ is a positive real number greater than 1. If z is a negative real number, $\exp(z) = 1/\exp(-z)$ is a positive real number less than 1.

For any complex number z,

$$\overline{\exp(z)} = \exp(\overline{z}), \tag{5.29}$$

where $\overline{a}$ denotes the complex conjugate of a complex number a. In particular, if $z = x + i\,y$, where x, y are real numbers, then

$$|\exp(z)| = \exp(x), \tag{5.30}$$

which is to say that

$$|\exp(i\,y)| = 1. \tag{5.31}$$

In fact,

$$\exp(i\,y) = \cos y + i\,\sin y, \tag{5.32}$$

as one can see from the power series expansions.

The exponential function is a holomorphic function of z whose complex derivative is equal to itself, just as the ordinary derivative of $\exp(x)$ for $x \in \mathbf{R}$ is equal to itself, as one can see from the power series. Note that one can also derive relations for the derivatives of the real and imaginary parts of $\exp(i\,y)$ as a function on the real line which are compatible with the formula in terms of sines and cosines above. The fact that the exponential of a sum is equal to the product of the individual exponentials also contains standard addition formulas for sines and cosines.

As a consequence of the trigonometric relations, we get that

$$\exp(2\pi\,i\,n) = 1 \tag{5.33}$$

when n is an integer, and that these are the only times when $\exp(z) = 1$. Also, $\exp(z) \neq 0$ for all $z \in \mathbf{C}$, and if ζ is a nonzero complex number, then there are z's in $\mathbf{C}$ such that

$$\exp(z) = \zeta. \tag{5.34}$$

Specifically, the real part x of z is uniquely determined by the condition that $\exp(x) = |\zeta|$, while the imaginary part y of z should be taken to be an "angle" for ζ, as in polar coordinates for $\mathbf{R}^2 \simeq \mathbf{C}$, which is determined up to adding integer multiples of 2π.

If U is a nonempty open subset of $\mathbf{C}\backslash\{0\}$ and $\lambda(\zeta)$ is a continuous complex-valued function on U, then we say that $\lambda(\zeta)$ is a *branch of the logarithm* on U if

$$\exp(\lambda(\zeta)) = \zeta \tag{5.35}$$

for all $\zeta \in U$. Assuming that $\lambda_1(\zeta)$, $\lambda_2(\zeta)$ are branches of the logarithm on the same nonempty open subset U of $\mathbf{C}\backslash\{0\}$, we get that

$$\lambda_1(\zeta) - \lambda_2(\zeta) \in 2\pi\,i\,\mathbf{Z} \tag{5.36}$$

for all $\zeta \in U$. Under the additional hypothesis that U is connected, it follows that there is a single integer n such that

$$\lambda_1(\zeta) - \lambda_2(\zeta) = 2\pi\,i\,n \tag{5.37}$$

As above, if $\lambda(\zeta)$ is a branch of the logarithm on a nonempty open subset U of $\mathbf{C}\backslash\{0\}$, then

$$\mathrm{Re}\,\lambda(\zeta) = \log|\zeta|, \tag{5.38}$$

where the logarithm $\log\rho$ of a positive real number ρ is the real number r characterized by $\exp(r) = \rho$. Also, $\lambda(\zeta)$ is a holomorphic function on U. The complex derivative of $\lambda(\zeta)$ is given by

$$\lambda'(\zeta) = \frac{1}{\zeta}. \tag{5.39}$$

The function $1/\zeta$, up to a nonzero constant multiple, is a fundamental solution for $\partial/\partial\overline{z}$ on $\mathbf{C}$. On $\mathbf{R}^n$ for general n one has the Clifford-valued function

$$\frac{\sum_{j=1}^n x_j\, e_j}{|x|^n}, \tag{5.40}$$

a nonzero constant multiple of which is a fundamental solution for $\mathcal{D}_L$, $\mathcal{D}_R$. Integrals of this function are also related to measurements of angular increments, as in the complex plane.

5.3 Linear algebra in $\mathbf{R}^n$, continued

Let U be a nonempty open subset of $\mathbf{R}^n$, and let $f(x)$ be a real-valued twice continuously-differentiable function on U. For each $x \in U$, we get the matrix of second partial derivatives of f,

$$\frac{\partial^2}{\partial x_j \partial x_l}, \quad 1 \le j, l \le n, \tag{5.41}$$

and a theorem of advanced calculus states that this matrix is symmetric. The Laplacian $\Delta f(x)$ of f at x is the same as the trace of this matrix.

If T is a linear transformation on $\mathbf{R}^n$ and v is a nonzero vector in $\mathbf{R}^n$, then v is said to be an *eigenvector* for T with *eigenvalue* $\lambda \in \mathbf{R}$ if

$$T(v) = \lambda\, v. \tag{5.42}$$

If T is a symmetric linear transformation on $\mathbf{R}^n$, and if v is an eigenvector of T with eigenvalue λ, then T maps every vector w in $\mathbf{R}^n$ orthogonal to v to a vector orthogonal to v. Indeed,

$$\begin{aligned} \langle T(w), v\rangle &= \langle w, T(v)\rangle \\ &= \lambda\,\langle w, v\rangle = 0. \end{aligned} \tag{5.43}$$

In general, a linear transformation T on $\mathbf{R}^n$ is said to be *diagonalizable* if there is a basis of $\mathbf{R}^n$ consisting of eigenvectors of T, which is equivalent to saying that there is an invertible linear transformation A on $\mathbf{R}^n$ so that the matrix associated to $A\,T\,A^{-1}$ is diagonal. If T is a symmetric linear transformation on

$\mathbf{R}^n$, then there is an orthonormal basis of $\mathbf{R}^n$ consisting of eigenvectors of T, which is equivalent to saying that there is an orthogonal linear transformation A on $\mathbf{R}^n$ such that the matrix associated to $A\,T\,A^{-1}$ is diagonal. To find an eigenvector v for T, one can maximize

$$\langle T(v), v\rangle \tag{5.44}$$

among vectors v such that $|v| = 1$, and then one can repeat the process to get an orthonormal basis of eigenvectors.

Suppose that T is any linear transformation on $\mathbf{R}^n$. Then $T^*\,T$ is a nonnegative symmetric linear operator on $\mathbf{R}^n$, and there is a nonnegative symmetric linear operator $\widehat{T}$ on $\mathbf{R}^n$ such that

$$\widehat{T}^2 = T. \tag{5.45}$$

Namely, there is an orthonormal basis of $\mathbf{R}^n$ consisting of eigenvectors of $T^*\,T$, with nonnegative eigenvalues, and one can define $\widehat{T}$ so that the same basis is a basis of eigenvectors and the eigenvalues are nonnegative square roots of those of T.

Observe that

$$|\widehat{T}(v)| = |T(v)| \tag{5.46}$$

for all $v \in \mathbf{R}^n$, since

$$\begin{aligned} |\widehat{T}(v)|^2 &= \langle \widehat{T}(v), \widehat{T}(v)\rangle \\ &= \langle \widehat{T}^2(v), v\rangle \\ &= \langle (T^*\,T)(v), v\rangle = \langle T(v), T(v)\rangle = |T(v)|^2. \end{aligned} \tag{5.47}$$

Using this one can show that there is an orthogonal linear transformation A on $\mathbf{R}^n$ such that

$$T = A \circ \widehat{T}. \tag{5.48}$$

This is called a *polar decomposition* of T, although one might prefer a decomposition of T in the other order, as the composition of a nonnegative symmetric operator with an orthogonal transformation, and this can be obtained by expressing T^* as the composition of an orthogonal transformation and a nonnegative symmetric operator, and taking the adjoint afterwards. For this polar decomposition, one can simply take $A = T\,\widehat{T}^{-1}$ when T is invertible, so that $\widehat{T}$ is invertible, and otherwise one has some freedom in choosing A on the orthogonal complement of the image of $\widehat{T}$.

Now let us look some more at orthogonal transformations in general. If $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are orthogonal bases on $\mathbf{R}^n$, then there is a unique linear transformation A on $\mathbf{R}^n$ such that $A(v_j) = w_j$ for $1 \le j \le n$, and A is an orthogonal transformation. Conversely, if A is an orthogonal linear transformation on $\mathbf{R}^n$ and $v_1, \ldots, v_n$ is an orthonormal basis of $\mathbf{R}^n$, then $A(v_1), \ldots, A(v_n)$ is also an orthonormal basis of $\mathbf{R}^n$.

If A is an orthogonal linear transformation on $\mathbf{R}^n$ and $v_1, \ldots, v_n$ is an orthonormal basis of $\mathbf{R}^n$, then A is almost determined by its values on $v_1, \ldots, v_{n-1}$.

Namely, $A(v_n)$ should be a unit vector orthogonal to $A(v_1), \ldots, A(v_{n-1})$, and there are only two choices for this vector, which are negatives of each other. If A is a rotation, which is to say an orthogonal linear transformation with determinant 1, so that A preserves orientations on $\mathbf{R}^n$, then A is determined by its values on $v_1, \ldots, v_{n-1}$.

Suppose that $u_1, \ldots, u_k$ are orthonormal vectors in $\mathbf{R}^n$ with $k \le n-2$, or simply that $n \ge 2$ if no such vectors are specified. If v, w are unit vectors in $\mathbf{R}^n$ which are orthogonal to $u_1, \ldots, u_k$, then one can make a continuous deformation of v to w through unit vectors in $\mathbf{R}^n$ which are orthogonal to $u_1, \ldots, u_k$. Indeed, there is a continuous deformation of the identity operator on $\mathbf{R}^n$ through orthogonal linear transformations on $\mathbf{R}^n$ which fixes u_j for $j = 1, \ldots, k$ and which deforms v to w.

Using this, one can show that any two rotations on $\mathbf{R}^n$ can be continuously deformed to each other through orthogonal linear transformations. For that matter, any two invertible linear transformations on $\mathbf{R}^n$ whose determinants have the same sign can be continuously deformed to each other. If A_1, A_2 are two invertible linear transformations on $\mathbf{R}^n$ whose determinants have opposite signs, then any continuous deformation from A_1 to A_2 through linear transformations on $\mathbf{R}^n$ must pass through at least one linear transformation with determinant.

Recall that a function $f(x)$ on $\mathbf{R}^n$ is said to be radial if it depends only on $|x|$. This is equivalent to saying that

$$f(A(x)) = f(x) \tag{5.49}$$

for all orthogonal linear transformations A on $\mathbf{R}^n$. For if x, y are elements of $\mathbf{R}^n$ such that $|x| = |y|$, then there is an orthogonal linear transformation A on $\mathbf{R}^n$ such that $A(x) = y$.

Suppose that U is a nonempty open subset of $\mathbf{R}^n$ and that f is a twice continuously-differentiable function on U. If A is an orthogonal linear transformation on $\mathbf{R}^n$, then $f \circ A$ is a twice continuously-differentiable function on $A^{-1}(U)$. One can check that the Laplacian of $f \circ A$ at a point $x \in A^{-1}(U)$ is equal to the Laplacian of f at $A(x)$.

In particular, $f \circ A$ is harmonic on $A^{-1}(U)$ if f is harmonic on U. On the complex plane, if ϕ is a holomorphic function on an open set V which takes values in another open subset U of $\mathbf{C}$, and if f is a harmonic function on U, then $f \circ \phi$ is a harmonic function on V. If f is also holomorphic, then $f \circ \phi$ is holomorphic on V. If $f(z)$ is a holomorphic function on an open subset U of $\mathbf{C}$, then

$$\widetilde{f}(z) = \overline{f(\overline{z})} \tag{5.50}$$

is a holomorphic function on $\overline{U} = \{\overline{z} : z \in U\}$.

Let A be an orthogonal linear transformation on $\mathbf{R}^n$ with associated matrix $(a_{j,k})$, so that the jth coordinate of $A(x)$ can be written as

$$\sum_{k=1}^{n} a_{j,k}\, x_k. \tag{5.51}$$

We can associate to A a linear isomorphism $\widehat{A}$ on the Clifford algebra $\mathcal{C}(n)$, defined by the conditions that $\widehat{A}(r) = r$ when r is a real number,

$$\widehat{A}(e_k) = \sum_{j=1}^{n} a_{j,k}\, e_j, \tag{5.52}$$

and of course $\widehat{A}$ acting on a product of e_l's is equal to the product of the corresponding $\widehat{A}(e_l)$'s. One can check that the assumption that A be an orthogonal transformation provides the appropriate compatibility conditions, namely, $\widehat{A}(e_l)^2 = -1$ for $l = 1, \ldots, n$ and $\widehat{A}(e_l)\,\widehat{A}(e_m) = -\widehat{A}(e_m)\,\widehat{(e_l)}$ when $l \neq m$, which basically means that these products have real part equal to 0.

Suppose that U is a nonempty open subset of $\mathbf{R}^n$, and that $f(x)$ is a $\mathcal{C}(n)$-valued continuously differentiable function on U, and let f_A denote the function on $A(U)$ defined by

$$f_A(x) = \widehat{A}(f(A^{-1}(x))). \tag{5.53}$$

One can check that $\mathcal{D}_L$ or $\mathcal{D}_R$ applied to f_A at a point $x \in A(U)$ is the same as $\widehat{A}$ applied to $\mathcal{D}_L$ or $\mathcal{D}_R$ of f, respectively, evaluated at $A^{-1}(x)$. In particular, f_A is left or right Clifford-holomorphic on $A(U)$ if f is on U.

It is easy to see from the definition of Clifford-holomorphicity that at each point in the domain of a Clifford-holomorphic function, the derivative of the function in the direction of one of the coordinates is determined by the derivatives of the function in the directions of the other coordinates. Using the invariance under orthogonal transformations described in the previous paragraph, one can check that the derivative of a Clifford-holomorphic function in any direction is determined by the derivatives in the orthogonal directions. Of course this is very clear in the classical case of holomorphic functions of a single complex variable.

We can also derive as follows. Let v be a vector in $\mathbf{R}^n$, $\widehat{v} = \sum_{j=1}^{n} v_j\, e_j$ the corresponding element of $\mathcal{C}(n)$, and $f(x)$ a $\mathcal{C}(n)$-valued continuously differentiable function on an open subset U of $\mathbf{R}^n$. If $\mathcal{D}_L f(x) = 0$ for some $x \in U$, then of course

$$\widehat{v} \sum_{l=1}^{n} e_l\, \frac{\partial}{\partial x_l} f(x) = 0, \tag{5.54}$$

which can be rearranged as an identity between the directional derivative of f at x in the direction of v and a combination of terms of the form $v_j\, \partial/\partial x_l f(x) - v_l\, \partial/\partial x_j f(x)$ times $e_j\, e_l$. These terms involve directional derivatives of f at x in directions orthogonal to v, and an analogous computation works for $\mathcal{D}_R f(x)\, \widehat{v}$ when $\mathcal{D}_R f(x) = 0$.

5.4 Some loose ends

Let U be a nonempty open subset of $\mathbf{R}^n$. If f_1, f_2 are two real-valued continuously differentiable functions with restricted support in U, consider the bilinear

form $B(f_1, f_2)$ acting on f_1, f_2 defined by

$$B(f_1, f_2) = \int_U \langle \nabla f_1(x), \nabla f_2(x) \rangle \, dx, \tag{5.55}$$

where $\nabla f(x)$ denotes the gradient of f at x, the vector whose components are the partial derivatives

$$\frac{\partial}{\partial x_j} f(x). \tag{5.56}$$

Of course $B(f_1, f_2)$ is symmetric in f_1, f_2, and if f_1 is also twice continuously-differentiable, we have that

$$B(f_1, f_2) = -\int_U \Delta f_1(x) \, f_2(x) \, dx. \tag{5.57}$$

Also notice that $B(f, f) \ge 0$, and indeed $B(f, f) = 0$ if and only if $f = 0$, since we are considering functions with restricted support.

One can extend this to consider functions which tend to 0 at the boundary of U and which have enough regularity, and it can be of interest to make some assumptions about U as well. In particular, for a nice domain U, one can try to diagonalize this bilinear form, and thus the Laplacian, with respect to the standard integral innner product

$$\int_U f_1(x) \, f_2(x) \, dx, \tag{5.58}$$

and for functions which tend to 0 at the boundary of U. This means looking for smooth functions f on U which are eigenfunctions of the Laplacian, so that there is a real number λ such that

$$\Delta f(x) = \lambda \, f(x) \tag{5.59}$$

for all $x \in U$, and which tend to 0 at the boundary of U.

Just as for harmonic functions, one can define what it means for a continuous function on U to be an eigenfunction for the Laplacian in the weak sense, and show that this implies that the function is smooth. There are also basic theories in analysis for showing that there are eigenfunctions of the Laplacian which tend to 0 at the boundary, and enough of them to diagonalize the Laplacian among functions which tend to 0 at the boundary. Of course there are some details in formulating this, and indeed some variations along these themes.

If $h(x)$ is a real-valued harmonic function on a nonempty open subset of $\mathbf{R}^n$, and if $\phi(x)$ is a real-valued continuously differentiable function with restricted support in U, then

$$\int_U |\nabla(h + \phi)(x)|^2 \, dx = \int_U |\nabla h(x)|^2 \, dx + \int_U |\nabla \phi(x)|^2 \, dx, \tag{5.60}$$

since

$$\int_U \langle \nabla h(x), \nabla \phi(x) \rangle \, dx = -\int_U \Delta h(x) \, \phi(x) \, dx = 0. \tag{5.61}$$

In particular, notice that

$$\int_U |\nabla(h+\phi)(x)|^2\,dx \ge \int_U |\nabla h(x)|^2\,dx. \tag{5.62}$$

More precisely, this works under suitable conditions for any function ϕ which vanishes at the boundary of U. Conversely, one can use this as a basis for looking for harmonic functions on U with given boundary values, by trying to minimize

$$\int_U |\nabla f(x)|^2\,dx \tag{5.63}$$

among functions with those boundary values.

Another approach to finding harmonic functions on a nonempty open subset U of $\mathbf{R}^n$ with specified boundary values is to consider the supremum of subharmonic functions on U which are less than or equal to the prescribed values at the boundary. A third method is to generate a lot of harmonic functions on U in general, and then choose ones with desired properties. We have already seen "fundamental solutions" on $\mathbf{R}^n \backslash \{0\}$ which are harmonic or holomorphic and hence harmonic, and one can use translates of these to get harmonic functions on $\mathbf{R}^n \backslash \{y\}$ for each y in $\mathbf{R}^n$, and then linear combinations of these with $y \in \mathbf{R}^n \backslash U$ to get harmonic functions on U.

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